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Scherk-Schwarz reduction of M-theory on G_2-manifolds with fluxes

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Abstract: We analyse the 4-dimensional effective supergravity theories obtained from the Scherk-Schwarz reduction of M-theory on twisted 7-tori in the presence of 4-form fluxes. We implement the appropriate orbifold projection that preserves a G_2-structure on the internal 7-manifold and truncates the effective field theory to an $\mathcal{N} = 1$, $D = 4$ supergravity. We provide a detailed account of the effective supergravity with explicit expressions for the Kähler potential and the superpotential in terms of the fluxes and of the geometrical data of the internal manifold. Subsequently, we explore the landscape of vacua of M-theory compactifications on twisted tori, where we emphasize the role of geometric fluxes and discuss the validity of the bottom-up approach. Finally, by reducing along isometries of the internal 7-manifold, we obtain superpotentials for the corresponding type IIA backgrounds.

Keywords: M-Theory, Superstring Vacua, Flux compactifications.

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Contents

1. Introduction

In the ordinary approach to 4-dimensional effective theories coming from flux compactifications, the effect of the fluxes is described by a mass deformation of the moduli fields. The moduli one considers are those of the special holonomy manifolds specifying the background at zero flux. This procedure obviously neglects the backreaction of the fluxes on the geometry and is based on the assumption that, for sufficiently "small" fluxes, the original moduli can still be used as the deformation degrees of freedom for the new backgrounds. This assumption is well justified in some cases, for example for type IIB compactifications on Calabi-Yau manifolds, where the only effect of the introduction of non-trivial 3-form fluxes is a conformal rescaling of the Calabi-Yau metric. There are, however, several examples of interesting geometries and models which cannot be described in this way, because the introduction of fluxes implies a drastic change in the topology of the underlying manifold and consequently of the moduli fields.

As of today we have exhaustive classifications of the differential and topological properties of the geometries that preserve some supersymmetry in the presence of fluxes. Unfortunately, very few explicit examples of manifolds satisfying the corresponding differential and topological constraints have been found so far. In this respect, twisted tori provide a simple, but already quite non-trivial, example of such geometries. From the point of view of connecting the properties of the effective theories to the geometry of the internal manifold they are even more valuable, because of their relation to Scherk-Schwarz reductions [1 − 3]. In the context of flux compactifications twisted tori were revived in [4, 5].

Flux compactifications on twisted tori are described by gauged supergravity models in four dimensions, where the moduli fields couple and receive masses from both the non-trivial connection of the internal manifold and the form fluxes. This fact has been crucial to obtain a simple type IIA model where all the bulk moduli are stabilized by using gauged supergravity techniques [6]. Although it is not yet clear whether this is really sufficient to stabilize all the moduli of the effective theory, there are by now several examples where this result is obtained by incorporating non-perturbative effects [7, 8], or by using a coset compactification of massive type IIA theory [9]. These examples show precisely the relevance of using the appropriate geometries for flux compactifications, as the introduction of fluxes is compensated by a twist in the geometry that changes the topological structure of the tori.

The moduli stabilization problem as well as any attempt to obtain some vacuum statistics depend heavily on the form of the (super)-potential of the effective theory. These potentials have a very simple form when Calabi-Yau or G_2 compactifications of string or M-theory are considered [10 − 13], while they become considerably more involved when the internal manifold is deformed away from special holonomy [14 − 17]. In these cases, the derivation of the (super)-potential is less clear-cut since it is based on several assumptions concerning the topology of the internal space. However, as suggested in [18], twisted tori compactifications can be used to derive such potentials in terms of the geometric structures of the internal manifold for more general cases. The type IIA case was analysed in detail in [6], providing an explicit form for the potential, which has indeed more general application [19]. Also, the possible group structures allowed for twisted tori compactifications in the type IIA case, as well as a more detailed analysis of the vacua and the corresponding moduli stabilization, were given in [20].

So far, we have a very good understanding of the Kähler potential describing the geometry of the moduli space as well as of the superpotential for type II and heterotic $\mathcal{N} = 1$ compactifications. Many insights on the corresponding M-theory objects have also been obtained [13, 21 − 23], but, as mentioned above, when the internal geometries are not given by special holonomy manifolds, the derivations are either incomplete or less rigorous.

For these reasons we focus here on $\mathcal{N} = 1$ compactifications of M-theory on twisted tori. The starting point is the $\mathbb{T}^7$ orbifolds of G_2-holonomy constructed by Joyce [24, 25] and some smooth generalizations thereof.[1] These models have 7 main moduli but once fluxes are turned on, be they of the 4-form field-strength or geometrical (i.e. Scherk-Schwarz deformations), the deformed backgrounds have less moduli, as some of the 7 light fields become massive. These deformed backgrounds, twisted $\mathbb{T}^7$, have no longer G_2-holonomy but rather G_2-structure. This implies that the effective theory still preserves $\mathcal{N} = 1$ supersymmetry at the lagrangian level.

We discuss the possible group structures that can be obtained by these geometric deformations and then determine which ones preserve some supersymmetry. For this purpose, we derive the form of the Kähler potential and superpotential of the effective theory and discuss its vacua, putting emphasis on the moduli stabilization problem. The Kähler potential and superpotential are expressed in a simple way in terms of the complexified G_2-structure $C + i\Phi$, where C is the 3-form gauge potential, and the 4-form background flux g:

$$e^{-K/3} = \frac{1}{7}\int_{X_7} \Phi \wedge \star\Phi, \qquad \mathcal{W} = \frac{1}{4}\int_{X_7} (C + i\Phi) \wedge \left[g + \frac{1}{2} d\,(C + i\Phi)\right]. \tag{1.1}$$

We will give this derivation in two different ways, providing also an extension of the pseudo-action description of [28] to M-theory.

The vacua analysis shows that supersymmetric critical points can be obtained only by using G_2-holonomy manifolds for Minkowski vacua or weak G_2-holonomy manifolds for AdS_4 vacua. As we will see, the twisted tori orbifolds we analyse here can only give the first type of supersymmetric vacua, while non-supersymmetric ones can be obtained by using different group structures.

A by-product of our study, using the so-called flat groups as playground, is the clarification of some aspects of Scherk-Schwarz reductions related to the precise definition of the moduli fields. In particular, we emphasize the difference between ordinary Kaluza-Klein compactification on a twisted torus and Scherk-Schwarz reduction.

Another important by-product of our analysis is the specification of the conditions that distinguish non-trivial vacua, i.e. vacua that correspond to distinct non-trivial backgrounds of the original 11-dimensional theory. We will actually see that most of the vacua that are obtained by a simple analysis of the effective theory represent the same compactification manifold using trivial cohomological transformations.

Finally, we consider the type IIA reduction of our backgrounds employing the construction of dynamic and fibered G_2-structures from 6-dimensional SU(3)-structure manifolds [29]. We show that the general expression for the type IIA superpotential found in the literature can be indeed recovered this way. Furthermore, we perform the reduction of the twisted $\mathbb{T}^7$ superpotentials and compare it with those of the corresponding type IIA

[1] The physics of the non-compact version of these backgrounds and other orbifold compactifications of M-theory was studied in [26]. For the compact models there exists an interesting type IIA orientifold interpretation [27].

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orientifold. The latter were constructed in [5] using gauged supergravity methods. This reduction shows that the M-theory superpotential has generically more quadratic couplings than the type IIA one. We discuss a possible geometrical origin of this discrepancy.

The layout of this paper is as follows. In section 2 we present the twisted 7-tori that will be the focal point of this work. In section 3 we find the 4-dimensional potential that describes the Scherk–Schwarz reduction of M-theory on the 7-torus including fluxes and we construct the corresponding superpotential. In section 4 we examine the vacuum structure of our (super)-potentials and address the issue of moduli stabilization for these backgrounds. In section 5 we perform the type IIA reduction of our results and compare it with those of [5]. Section 6 summarizes our findings and discusses potential future directions. Finally, in the appendix, we show how the pseudo-action that we use to derive the scalar potential of the effective supergravity theory can be utilized to describe the general gauged supergravity algebras of Scherk–Schwarz M-theory compactifications on $\mathbb{T}^7$ in terms of the "dual" degrees of freedom, avoiding any recourse to Free Differential Algebras that appear when massive tensor fields survive in the effective theory.

2. The model: twisted 7-torus

In this section we present our model and discuss some of its geometrical features that will be relevant for our analysis. The basic idea is to twist the toroidal orbifolds of [24, 25] away from G_2-holonomy, therefore obtaining a 7-manifold with G_2-structure. Consistency of the orbifold action with the twisting eliminates some of the possible twists and demands that the G_2-structure is cocalibrated, i.e. has a coclosed associative 3-form. The moduli are introduced in analogy with the toroidal case and according to the Scherk-Schwarz ansatz [2, 3].

2.1 $\mathbb{T}^7$ orbifolds with G_2-holonomy

Our starting point is the compact G_2-holonomy[2] manifolds obtained as toroidal orbifolds of the form $X_7 = \mathbb{T}^7/(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ [24, 25]. Let us take $y^I,\ I = 5, \ldots, 11$ as coordinates on $\mathbb{T}^7$. Then, the $(\mathbb{Z}_2)^3$ action is

$$\begin{aligned}
\mathbb{Z}_2(y^I) &= \{-y^5, -y^6, -y^7, -y^8, c + y^9, y^{10}, y^{11}\}\,,\\
\mathbb{Z}_2'(y^I) &= \{a_1 - y^5, a_2 - y^6, y^7, y^8, c - y^9, c - y^{10}, c + y^{11}\}\,,\\
\mathbb{Z}_2''(y^I) &= \{a_3 - y^5, y^6, a_4 - y^7, y^8, a_5 - y^9, c + y^{10}, -y^{11}\}\,,
\end{aligned} \tag{2.1}$$

where the coefficients a_i and c can be either 0 or 1/2. This orbifold has singularities that can be either blown-up or eliminated by turning on c and a_4 simultaneously. Then the orbifold action becomes free and produces a smooth 7-manifold. It is easy to show that the three $\mathbb{Z}_2$ groups square to the identity and commute. The (untwisted) Betti numbers of X_7 are independent of a_i and c; they are given by $b_1(X_7) = 0$, $b_2(X_7) = 0$, $b_3(X_7) = 7$. Notice

[2] To be precise, these orbifolds have a discrete holonomy group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \subset G_2$. Only the manifolds obtained after blowing up the singularities have G_2-holonomy.

that resolving the singularities yields extra harmonic forms and the Betti numbers change. Here, we will consider only the bulk (i.e. untwisted) cohomologies and the corresponding geometrical and physical moduli.

The orbifold action (2.1) preserves a G_2-structure, given by the G_2 invariant combination of the 7 surviving 3-forms. In terms of vielbeins $e^I = R_I dy^I$, $I = 5, \dots, 11$, the associated 3-form Φ reads

$$\Phi = e^5 \wedge e^6 \wedge e^{11} - e^7 \wedge e^8 \wedge e^{11} - e^9 \wedge e^{10} \wedge e^{11} + e^5 \wedge e^7 \wedge e^{10} - e^6 \wedge e^7 \wedge e^9 + e^5 \wedge e^8 \wedge e^9 + e^6 \wedge e^8 \wedge e^{10} . \tag{2.2}$$

This form is obviously closed and coclosed, which implies that the holonomy group of X_7 is contained in G_2. Actually, because of the orbifold projections, this is the only 3-form (up to signs) that is G_2-invariant, and therefore the holonomy group is exactly G_2. Consequently, the effective 4-dimensional theory obtained by compactification on X_7 can be recast in the form of an $\mathcal{N} = 1$ supergravity coupled to matter. This can also be verified by computing the spectrum of the surviving fields.

Using the information from the Betti numbers, we deduce that the effective theory for the bulk fields contains only the $\mathcal{N} = 1$ graviton multiplet and 7 chiral multiplets T_i. This follows from the generic decomposition of the M-theory 3-form:

$$C = A^\alpha(x) \wedge \omega_\alpha(y) + \tau_i(x) \phi^i(y) , \tag{2.3}$$

where ω_α and ϕ_i span $H^2(X_7)$ and $H^3(X_7)$ respectively, and from the surviving metric components:

$$g_{\mu\nu}, \quad g_{II} . \tag{2.4}$$

Since $b_2(X_7) = 0$ there are no vector multiplets. The chiral multiplets contain the moduli of the theory parametrizing the coset $[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}]^7$. The imaginary parts τ_i of these moduli come from the reduction (2.3) of the M-theory 3-form potential on the internal manifold:

$$\begin{aligned} C = & -\tau_1 dy^5 \wedge dy^6 \wedge dy^{11} + \tau_2 dy^7 \wedge dy^8 \wedge dy^{11} + \tau_3 dy^9 \wedge dy^{10} \wedge dy^{11} + \tau_4 dy^6 \wedge dy^7 \wedge dy^9 - \\ & - \tau_5 dy^6 \wedge dy^8 \wedge dy^{10} - \tau_6 dy^5 \wedge dy^7 \wedge dy^{10} - \tau_7 dy^5 \wedge dy^8 \wedge dy^9 . \end{aligned} \tag{2.5}$$

The real parts t_i are associated with the internal metric components and parametrize the volumes of the seven surviving 3-cycles:

$$\Phi = t_i(x) \phi^i(y) . \tag{2.6}$$

The holomorphic combinations $T_i = t_i + i\,\tau_i$ can be thought of as expansion parameters of a "complexified" G_2-form

$$C + i\,\Phi = i\,T_i(x)\,\phi^i(y) . \tag{2.7}$$

This is analogous to the complexification of Kähler moduli in type II string theories.

In terms of the radii R_I of the 7 circles composing $\mathbb{T}^7$, we have $e^I = R_I dy^I$ and hence

$$\begin{aligned} &|t_1| = R_5 R_6 R_{11} , \quad && |t_2| = R_7 R_8 R_{11} , \quad && |t_3| = R_9 R_{10} R_{11} , \quad && |t_4| = R_6 R_7 R_9 , \\ &|t_5| = R_6 R_8 R_{10} , && |t_6| = R_5 R_7 R_{10} , && |t_7| = R_5 R_8 R_9 . && \end{aligned} \tag{2.8}$$

We can also write the vielbeins in terms of the moduli as

$$
\begin{aligned}
e^5 &= \sqrt{\frac{t_1 t_6 t_7}{V}} dy^5, \qquad e^6 = \sqrt{\frac{t_1 t_4 t_5}{V}} dy^6, \qquad e^7 = \sqrt{\frac{t_2 t_4 t_6}{V}} dy^7, \\
e^8 &= \sqrt{\frac{t_2 t_5 t_7}{V}} dy^8, \qquad e^9 = \sqrt{\frac{t_3 t_4 t_7}{V}} dy^9, \qquad e^{10} = \sqrt{\frac{t_3 t_5 t_6}{V}} dy^{10}, \\
e^{11} &= \sqrt{\frac{t_1 t_2 t_3}{V}} dy^{11}, \qquad V = (t_1 \cdots t_7)^{1/3}.
\end{aligned} \tag{2.9}
$$

We need to impose the conditions $t_i \neq 0$, $\forall i$, so as to avoid metric degeneracies. Furthermore, the combinations under the square roots in (2.9) must be positive (which means that either all the t_i are positive or four of them are negative and two positive).

2.2 Twisting and G$_2$-structures

We now consider the orbifold action (2.1) on a twisted $\mathbb{T}^7$. Physically, the twisting can be interpreted as performing a Scherk–Schwarz reduction on the *original* untwisted torus. Practically, it simply means that we have to replace the straight differentials dy^I in (2.9) by twisted ones, i.e. 1-forms η^I that satisfy

$$d\eta^I - \frac{1}{2}\tau^I_{JK}\eta^J \wedge \eta^K = 0 \tag{2.10}$$

with structure-constant parameters $\tau^I_{JK} = -\tau^I_{KJ}$ [3, 18, 31, 6].

For a generic Scherk–Schwarz reduction, these constants are constrained by

$$\tau^I_{IJ} = 0\,, \qquad \tau^L_{[IJ}\tau^M_{K]L} = 0\,. \tag{2.11}$$

The first condition guarantees that we can integrate out the dependence on the coordinates of the internal manifold and obtain a lagrangian 4-dimensional theory. This condition can be relaxed when we consider a reduction of the equations of motion without requiring a well-defined lagrangian. The second condition stems from the nilpotency of exterior differentiation and therefore it is relevant when we want to interpret our Scherk–Schwarz reduction as a compactification on a twisted torus. Furthermore, the consistency of (2.11) with the orbifold action implies that we can keep only those structure constants τ^I_{JK} with indices (I, J, K) acted upon as $(+,+,+), (+,-,-), (-,-,+), (-,+,-)$ under (2.1). Hence, the only surviving structure constants are

$$\left\{\tau^7_{510}, \tau^5_{710}, \tau^{10}_{57}\right\}, \ \left\{\tau^6_{79}, \tau^7_{69}, \tau^9_{67}\right\}, \ \left\{\tau^8_{59}, \tau^9_{58}, \tau^5_{89}\right\}, \ \left\{\tau^{10}_{68}, \tau^8_{610}, \tau^6_{810}\right\}, \tag{2.12}$$

$$\left\{\tau^{11}_{56}, \tau^{11}_{78}, \tau^{11}_{910}\right\}, \tag{2.13}$$

$$\left\{\tau^5_{611}, \tau^6_{511}, \tau^7_{811}, \tau^8_{711}, \tau^9_{1011}, \tau^{10}_{911}\right\}. \tag{2.14}$$

Notice that the first condition in (2.11) is automatically satisfied by the surviving structure constants.

When the twist in the metric is introduced, the space X_7 is no longer a G$_2$-holonomy manifold, but its tangent bundle rather shows a G$_2$-structure. This means that the rank-3 invariant form in (2.2), written now in terms of the twisted vielbeins $e^I = R_I\ \eta^I$, is preserved by the orbifold action and is a singlet of the group structure of the tangent bundle. However, a priori Φ is no longer closed and coclosed; the deviation from special holonomy is parametrized by 4 intrinsic torsion classes W_1, W_7, W_{14}, W_{27}, which are defined as

$$d\Phi = W_1 \star\Phi - \Phi \wedge W_7 + W_{27}\,, \tag{2.15}$$

$$d\star\Phi = \frac{4}{3}\star\Phi \wedge W_7 + W_{14}\,, \tag{2.16}$$

and where $\Phi \wedge W_{27} = 0$ and $\Phi \lrcorner W_{27} = 0$. The subscripts refer to the representations of the various components under G_2. Note that the Betti numbers of the twisted tori depend on the specific set of structure constants we choose to turn on and in general are different than those of the original untwisted $\mathbb{T}^7$.

It is interesting to notice that not all of these torsion classes can be realized using twisted tori. By direct inspection of the exterior differentials of Φ and its dual, one can actually find that the coassociative form defined on X_7 is closed for the twists (2.14) that survive the orbifold projection:

$$W_7 = W_{14} = 0\,. \tag{2.17}$$

Hence, the allowed G_2 structures are cocalibrated [32], i.e. they satisfy

$$d\Phi = W_1 \star\Phi + W_{27}\,, \tag{2.18}$$

$$d\star\Phi = 0\,. \tag{2.19}$$

W_1 and W_{27} can be explicitly computed in terms of the geometric fluxes coefficients. For example (setting all the radii to 1)

$$\begin{aligned} W_1 = \frac{2}{7}\Big(&\tau^5_{611} - \tau^8_{59} - \tau^8_{610} - \tau^9_{1011} - \tau^9_{67} + \tau^8_{711} + \tau^9_{58} + \tau^{10}_{911} + \tau^{10}_{57} + \tau^{10}_{68} - \tau^{11}_{78} + \\ &+ \tau^5_{710} + \tau^{11}_{56} - \tau^{11}_{910} + \tau^5_{89} - \tau^6_{511} - \tau^6_{79} + \tau^6_{810} - \tau^7_{811} - \tau^7_{510} + \tau^7_{69}\Big)\,, \end{aligned} \tag{2.20}$$

while the 4-form W_{27} can be also easily obtained from (2.18). Obviously, not all of these torsions will allow for supersymmetric vacua. We will see later, in the analysis of the vacua of the superpotential, which examples are concretely realized out of the possible ones described here.

We should stress here that the manifold under consideration has precisely a G_2-structure and not subgroups thereof, since 1- and 2-forms are projected out by the orbifold. Furthermore, the fact that the G_2-structure we obtain is coclosed is also a consequence of the orbifold projections. A generic twisted 7-torus can actually contain all the torsion components in (2.15) and (2.16). It is also clear that this more generic situation would allow extended 4-dimensional supersymmetries; the analysis should therefore be refined by using SU(3) or even SU(2) structures.

3. Potential and superpotential

In order to derive the effective potential and superpotential in the presence of both form and geometrical fluxes, we have to reduce the action of 11-dimensional supergravity on X_7. For what concerns $\mathcal{N} = 1$ effective theories, various derivations of the superpotential have

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been presented either for G_2-holonomy manifolds with 4-form flux turned on [12, 13, 33–35], or by using weak-G_2-holonomy manifolds [22, 21], or by reducing the supersymmetry transformations [36, 37]. In the following we will extend these results for twisted tori compactifications.

The reduction of M-theory in the presence of 4-form fluxes G_4 is subtle, since the Chern–Simons terms are modified in the presence of topologically non-trivial background flux g. It was shown in [7] that invariance under large gauge transformations fixes completely the Chern–Simons terms and the correct action is

$$S_{11d} = \frac{1}{2}\int d^{11}x\sqrt{-g}R - \frac{1}{4}\int (g+dC)\wedge\star(g+dC) - \frac{1}{12}\int dC\wedge dC\wedge C - \\ - \frac{1}{4}\int dC\wedge g\wedge C - \frac{1}{4}\int g\wedge g\wedge C\,, \tag{3.1}$$

where the standard 4-form combination $G_4 = g + dC$ appears only in the kinetic term, while the Chern–Simons terms cannot be recast in terms of only C and G_4. These non-trivial Chern-Simons terms appear only when one considers fluxes g that are closed but not exact. Obviously the exact part will still appear with the same coefficient as dC. Notice that varying this action with respect to the 3-form gauge tensor C the equation of motion for G_4 is obtained, which again depends only on the expected combination $G_4 = g + dC$.

The reduction of the Einstein kinetic term was first performed in the seminal paper by Scherk and Schwarz [2]; with our normalizations, it reads

$$V_E = \frac{1}{8}\frac{1}{e_7}\left(2\,\tau^I_{JK}\tau^J_{IL}g^{KL} + \tau^I_{JK}\tau^L_{MN}g_{IL}g^{JM}g^{KN}\right). \tag{3.2}$$

The volume of the internal space $e_7 = (t_1\ldots t_7)^{1/3}$ appears in the denominator because of the rescaling of the space–time metric $g_{\mu\nu} \to e_7\, g_{\mu\nu}$ required to go to the Einstein frame in 4-dimensions.

Given the expansion of the 4-form field strength vacuum expectation value

$$g = \frac{1}{4!}g_{IJKL}\eta^I\wedge\eta^J\wedge\eta^K\wedge\eta^L\,, \tag{3.3}$$

which is the appropriate one for a Scherk-Schwarz reduction with fluxes [3], the 4-form kinetic term yields

$$V_K = \frac{1}{96}\frac{1}{e_7}\left(g_{IJKL} + 6\,\tau^P_{[IJ}C_{KL]P}\right)g^{II'}g^{JJ'}g^{KK'}g^{LL'}\left(g_{I'J'K'L'} + 6\,\tau^{P'}_{[I'J'}C_{K'L']P'}\right). \tag{3.4}$$

As a consequence of the Bianchi identity, the topological flux g must obey

$$\tau^P_{[IJ}g_{JKL]P} = 0\,. \tag{3.5}$$

Actually, for our model these conditions are identically satisfied, thanks to the orbifold projections.

Finally, the Chern-Simons contribution to the scalar potential can be obtained by integrating the 4-form with external legs, which acts as a Lagrange multiplier: $G_{\mu\nu\rho\sigma} = \lambda\epsilon_{\mu\nu\rho\sigma}$. By doing so one obtains

$$V_{\rm CS} = \frac{1}{3!4!}\frac{1}{e_7}\left[\frac{1}{4!e_7}\epsilon^{IJKLMNP}\left(g_{IJKL} + 3\,\tau^Q_{[IJ}C_{KL]Q}\right)C_{MNP}\right]^2. \tag{3.6}$$

What should be noted in this expression is the factor of 2 difference with the similar term that comes from the ordinary G_4 combination appearing in V_K. This difference is due to the non-standard form of the Chern–Simons terms, which guarantees gauge invariance under large gauge transformations. This difference actually is crucial in order to express the potential in terms of the superpotential. Again, this happens only when considering fluxes g, which are closed but not exact. Any exact part can be reabsorbed in the potential as a constant background value for the C_{IJK} scalars and appears with the same coefficients as the parts involving τC. From this fact we can also understand that $G_4 = 0$ will not be a critical point of the potential whenever the non-exact part is non-vanishing $g \neq 0$ (we will show later that the vacua in [38] are trivial redefinitions of the zero-flux vacuum).

The derivation performed so far is clearly general and gives a potential for an $\mathcal{N} = 8$ effective supergravity.[3] However, thanks to the orbifold projections, the sum of the various terms can be recast in the standard $\mathcal{N} = 1$ form without D-terms

$$V = e^K \left(g^{i\bar{\jmath}} D_i \mathcal{W} D_{\bar{\jmath}} \overline{\mathcal{W}} - 3|\mathcal{W}|^2 \right), \tag{3.7}$$

where $D_i\mathcal{W} = \partial_i \mathcal{W} + \partial_i K\, \mathcal{W}$. An explicit calculation shows that this happens using the Kähler potential of the $[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}]^7$ scalar manifold:

$$K = -\sum_{i=1}^{7} \log \frac{T_i + \overline{T}_i}{2} = -\log\left(t_1 t_2 t_3 t_4 t_5 t_6 t_7\right), \tag{3.8}$$

and the superpotential

$$\begin{aligned} 4\,\mathcal{W} = &-T_5 T_7\, \tau^{8}_{711} - T_6 T_7\, \tau^{5}_{611} + T_4 T_7\, \tau^{9}_{1011} \\ &+ T_1 T_2\, \tau^{11}_{910} + T_1 T_3\, \tau^{11}_{78} - T_2 T_3\, \tau^{11}_{56} - T_1 T_4\, \tau^{6}_{810} + T_2 T_4\, \tau^{7}_{510} - T_3 T_4\, \tau^{9}_{58} + \\ &+ T_1 T_5\, \tau^{6}_{79} + T_2 T_5\, \tau^{8}_{59} - T_3 T_5\, \tau^{10}_{57} + T_4 T_5\, \tau^{6}_{511} - T_1 T_6\, \tau^{5}_{89} - T_2 T_6\, \tau^{7}_{69} \\ &+ T_4 T_6\, \tau^{7}_{811} - T_3 T_6\, \tau^{10}_{68} - T_5 T_6\, \tau^{10}_{911} - T_1 T_7\, \tau^{5}_{710} + T_2 T_7\, \tau^{8}_{610} + T_3 T_7\, \tau^{9}_{67} + \\ &+ \mathrm{i}\big(T_1\, g_{78910} - T_2\, g_{56910} - T_3\, g_{5678} - T_4\, g_{581011} + T_5\, g_{57911} + \\ &\qquad + T_6\, g_{68911} + T_7\, g_{671011}\big)\,. \end{aligned} \tag{3.9}$$

The naïve reduction generically produces terms that could not be recast in the $\mathcal{N} = 1$ form (3.7), but which vanish once the orbifold projections and the constraints (2.11) on the torsions are taken into account. In order to really consider only the topological flux contributions in (3.9), one should take into account additional constraints on the allowed values of the flux components, due to the topology changes driven by the twistings τ_{IJ}^{K}. For this reason we will allow generic values and then distinguish the cohomologically different parts.

The potential obtained by plugging (3.9) in (3.7) does not contain the possible non-trivial cosmological constant contributions that are due to expectation values of the 4-form on the external part. In our framework, since we would like to have a complete control

[3] While this paper was in preparation, a similar result was also obtained in [38] by integrating the equations of motion of 11-dimensional supergravity.

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also on this value, the cosmological term is best described by the expectation value of the dual 7-form $\tilde{G}_7$. This dual form obviously needs a different 11-dimensional formulation from (3.1). The mechanism we decided to use is the M-theory generalization of the pseudo-action approach, which allows for a dual formulation of type IIA supergravity [28] and which was also used in a similar reduction in [6]. This mechanism requires the formulation of a pseudo-action where the potentials and the dual curvatures appear. The first are considered as real degrees of freedom, while the dual curvatures are just "empty boxes", which act as multiplicative constants. Then, by varying the pseudo-action with respect to the potentials, one obtains Bianchi identities for the dual curvatures, which can be solved in terms of dual potentials. Finally, these solutions are plugged back in the pseudo-action, giving a real action to use for deriving the equation of motion of the potentials corresponding to the curvatures in the pseudo-action.

Trying to adapt this procedure in M-theory, we realize that, because of the Chern-Simons terms, the expected (dual) action for 11-dimensional supergravity cannot be used if we want to describe all the non-metric degrees of freedom[4] in terms of the standard potential C or of the dual one A. However, if one is interested in a split description of the potential degrees of freedom and of their curvatures, partly by the 3-form and partly by the 6-form, this approach is effective, as we will see in a moment.

The starting point is the pseudo-action

$$\begin{aligned} S_{11d} = \frac{1}{2}\int d^{11}x\sqrt{-g}R - \frac{1}{4}\int G_4 \wedge \star G_4 - \frac{1}{4}\int \tilde{G}_7 \wedge \star\tilde{G}_7 - \\ - \frac{1}{12}\int C \wedge \left(d\tilde{G}_7 + G_4 \wedge G_4\right) - \int A \wedge dG_4\,, \end{aligned} \tag{3.10}$$

where only some couples $\{C, \tilde{G}\}$ and $\{A, G\}$ are kept and G_4, $\tilde{G}_7$ are empty boxes for the moment. For the case at hand, the orbifold projection leaves little choice, since the only components of the potentials and fluxes that survive are

$$\begin{aligned} &C_{\mu\nu\rho}, \quad C_{IJK}, \quad A_{\mu\nu\rho IJK}, \quad A_{\mu\nu IJKL}\,, \\ &G_{\mu\nu\rho\sigma}, \quad G_{\mu IJK}, \quad G_{IJKL}, \quad \widetilde{G}_{\mu\nu\rho\sigma IJK}, \quad \widetilde{G}_{\mu\nu\rho IJKL}, \quad \widetilde{G}_{IJKLMNP}\,. \end{aligned} \tag{3.11}$$

Since we want to obtain a final action that depends on C_{IJK} and contains the curvatures G_{IJKL}, $G_{\mu IJK}$, $G_{IJKLMNP}$, we start with (3.10), where we set to zero all the terms containing this potential, as well as all the curvatures dual to the ones we want to appear in the final action. This means that we set to zero $G_{\mu\nu\rho\sigma}$, $\tilde{G}_{\mu\nu\rho\sigma IJK}$ and $\tilde{G}_{\mu\nu\rho IJKL}$. By doing so, (3.10) contains only 3 potential terms, depending on $C_{\mu\nu\rho}$, $A_{\mu\nu IJKL}$ and $A_{\mu\nu\rho IJK}$. The variation with respect to these components gives the equations of motion:

$$\partial_\mu \tilde{G}_{IJKLMNP} + 70\, G_{\mu[IJK}G_{LMNP]} = 0\,, \tag{3.12}$$

$$\partial_{[\mu}G_{\nu]IJK} = 0\,, \tag{3.13}$$

$$\partial_\mu G_{IJKL} - 6\,\tau_{[IJ}^{M} G_{\mu KL]M} = 0\,, \tag{3.14}$$

which can be interpreted as Bianchi identities for the various curvatures.

[4] An alternative formulation that allows for this description is given by the duality symmetric actions constructed in [39, 40].

Solving the Bianchi identities (3.12)–(3.14) we obtain the definitions of these curvatures in terms of the basic degrees of freedom and background fluxes on the internal manifold (we do not allow for constant background values of the curvatures with mixed indices):

$$G_{\mu IJK} = \partial_\mu C_{IJK}\,, \tag{3.15}$$

$$G_{IJKL} = g_{IJKL} + 6\tau^M_{[IJ}C_{KL]M}\,, \tag{3.16}$$

$$\tilde{G}_{IJKLMNP} = g_{IJKLMNP} - 70\left(g_{[IJKL} + 3\tau^Q_{[IJ}C_{KL|Q|}\right)C_{MNP]}\,. \tag{3.17}$$

The first curvature definition is a trivial solution to (3.13), and (3.16) follows from (3.14) when plugging the solution (3.15) in it. The last one is a bit more tricky, as the equation (3.12) does not become a total space–time derivative of a single quantity, even after plugging the solution (3.15) and (3.16) in it. However, inspection of the orbifold conditions shows that $\tau\,(C \wedge \partial_\mu C) = 0$, which implies that one can rewrite (3.12) as a space–time derivative on a single object.

Plugging these solutions back in the pseudo-action (3.10) results in a real action, which gives a 4-dimensional potential with precisely the various terms computed above. In addition, the Chern–Simons term is now added by a constant shift, which depends on the 7-form flux $g_{IJKLMNP}$. It is also interesting to notice that from this approach we do not obtain the 4-form Bianchi identity (3.5). This, however, is identically satisfied with our orbifold projections and therefore does not impose additional constraints. Also notice the factor of 2 between the constant 4-form flux and the cohomologically trivial part in (3.16) and (3.17). These are precisely the same factors which enter in (3.4) and (3.6) respectively and for which we have already given a detailed account.

We emphasize here that this framework is the most appropriate for studying the effect of M5-branes wrapped on 2-cycles of the internal manifold in the above setup, since these naturally couple to the 6-form flux A. For the orbifold at hand, however, we can easily see that the $A_{\mu\nu\rho\sigma IJ}$ component is projected out since there are no surviving 2-cycles and hence one cannot introduce this type of M5-branes. In more general cases, introducing these wrapped M5-branes will affect the constraints on the structure constants of the effective gauge couplings in 4-dimensions. More specifically, since the Bianchi identity of the 4-form flux will now be modified by a source term, this will imply a different closure of the gauge algebra governing the lower dimensional effective theory.

We are now in a position to write the Kähler potential and the superpotential in terms of basic geometrical quantities, using the definitions (2.2) and (2.5) and with the straight differentials dy^I being substituted by the 1-forms η^I. The Kähler potential reads

$$K = -3\log\left(\frac{1}{7}\int_{X_7}\Phi \wedge \star\Phi\right), \tag{3.18}$$

and the superpotential is given by

$$\mathcal{W} = \frac{1}{4}\int_{X_7}\tilde{G}_7 + \frac{1}{4}\int_{X_7}(C + i\,\Phi) \wedge \left[g + \frac{1}{2}d\,(C + i\,\Phi)\right], \tag{3.19}$$

where the exterior differentiation on the internal space satisfies (2.10) and gives rise to the terms containing the structure constants τ^K_{IJ}. The Kähler potential (3.18) coincides with the Hitchin functional describing the space of stable 3-forms for 7-manifolds. In analogy to type II compactifications on SU(3)-structure manifolds [41], it would be interesting to prove that this functional describes the moduli space of generic G_2-structure deformations, as suggested by (3.18).

Some comments are now in order concerning the general form of the superpotential. First of all, note that the flux contributes with terms linear in the holomorphic coordinates while the torsion is coupled to a quadratic form of the coordinates. Owing to the structure of the C form, analogous to Φ, we also see that only the W_1 torsion class computed in (2.20) contributes to $\mathcal{W}$. For the same reason, the 4-form flux contributes only with its singlet part. This is expected since invariance of $\mathcal{W}$ demands that only singlets of G_2 should appear in it. In the absence of a warp factor, the vanishing of this superpotential, which is a necessary condition for supersymmetric Minkowski vacua, implies that these components vanish, in accordance with the analysis of the supersymmetry variations [42, 43, 36].

Secondly, we believe that such an expression has more general validity than the twisted tori compactifications used here for its derivation. The superpotential for G_2-holonomy manifolds with 4-form flux turned on was computed in [13], while [21] extended this result to the case of G_2 structure manifolds. In the latter work, the potential was computed from explicit compactification only for manifolds with weak G_2 holonomy, i.e. for manifolds with only W_1 non-vanishing. Our derivation here for the twisted 7-torus extends this result to a situation where both W_1 and W_{27} are different from zero and is a further consistency check on the proposal of [21]. We should mention that the comparison with [13] in the limit of G_2 holonomy manifolds shows a different factor; this, however, comes from the assumption in [13] that dC still gives a non-vanishing contribution to the potential, and it can be reconciled with that of [13] after an integration by parts [22].

Furthermore, although our expression for the superpotential contains the same terms as those in [21], there is a clear difference between us and [13] on one side and [21] on the other. The difference is in the distribution of the various terms in the real and imaginary components of $\mathcal{W}$ and is actually crucial for deriving supersymmetric vacua.

4. Vacua and interpretation

As we stated in the introduction, the main purpose of constructing the effective theory for M-theory Scherk–Schwarz compactifications with fluxes is to provide an alternative way of determining supersymmetric backgrounds and studying the landscape of vacua when the internal manifolds are given by twisted tori. Hopefully this can help generalizing the results of [44] when the internal manifolds have non-trivial intrinsic torsion. In order to do so, the effective theory should capture all the properties of consistent 11-dimensional vacua. A very important consequence of this fact is that we should not expect critical points of the potential corresponding to Minkowski (or de Sitter) vacua with non-trivial fluxes. Instead, an AdS vacuum may be in principle possible. These expectations are due to the no-go theorem of [45, 46].

Under quite general assumptions, the only allowed compactifications of M-theory to 4-dimensions are either Minkowski vacua with $G_4 = g_7 = 0$ or Freund-Rubin solutions, giving AdS_4 with $G_4 = 0$ but $g_7 \neq 0$. Allowing for the presence of source terms like M-branes, may lead to more general types of supersymmetric configurations, possibly with non-trivial 4-form flux. Another way of bypassing this no-go theorem is the addition of higher order derivative terms in the action, for example terms involving higher powers of the Riemann curvature tensor. Since in the following we will stick to the setup described in section 3, where our starting point is pure 11-dimensional supergravity without higher order corrections, we should expect that our results be in accord with the no-go theorem and its implications.

4.1 General properties

As a first step for the analysis of the vacua, we discuss some generic features of the superpotentials of the form (3.9). We use a compact version of (3.9), which can be written as

$$\mathcal{W} = \frac{1}{2} M_{ij} T^i T^j + i\, G_i T^i + g_7\,, \tag{4.1}$$

where M, G and g_7 are all real and are associated to the geometrical, the 4-form, and the 7-form fluxes respectively. Moreover, the matrix M is symmetric, with zeros along the diagonal. In this way we can discuss the critical points without reference to a particular configuration of geometric and/or form fluxes. A similar expression has also appeared in [21] but for the crucial difference in the factor of i in front of the 4-form fluxes.

The general form (4.1) can be easily read from (3.9), with the addition of the 7-form flux, but it is also useful to derive it directly from (3.19). The latter method has the advantage of producing a compact expression for M (see also [21]) and can be used to formulate the supersymmetry conditions in terms of the torsion classes in (2.15) and (2.16). For this purpose, we use a basis of 3-forms ϕ^i (which coincides with the basis of seven harmonic 3-forms of the untwisted manifold when the structure constants are set to zero) and dual 4-forms $\widetilde{\phi}^i$ satisfying[5]

$$\int_{X_7} \phi^i \wedge \widetilde{\phi}^j = \delta^{ij}\,. \tag{4.2}$$

Plugging in (3.19) the expansions $C + i\Phi = i\, T_i\, \phi^i$ and $g_4 = 4 G_i \widetilde{\phi}^i$ yields

$$4\mathcal{W} = \frac{1}{2} T_i T_j \int_{X_7} \phi^i \wedge d\phi^j + 4i\, T_i G_j \int_{X_7} \phi^i \wedge \widetilde{\phi}^j + \int_{X_7} \widetilde{G}_7\,. \tag{4.3}$$

Using (4.2) and identifying

$$g_7 = \int_{X_7} \widetilde{G}_7 \tag{4.4}$$

and

$$M_{ij} = -\frac{1}{4} \int_{X_7} \phi^i \wedge d\phi^j\,, \tag{4.5}$$

gives (4.1). By definition M_{ij} is symmetric and $M_{ii} = 0$, $\forall i$. For our setup of orbifolds of twisted tori, the expansion of the differential of the basis of 3-forms is

$$d\phi^i = -4 M_{ij} \widetilde{\phi}^j. \tag{4.6}$$

[5]The 4-forms are not directly expanded using the $\star\phi^i$ forms, which depend explicitly on the metric, but rather in the $\widetilde{\phi}^i$ basis, which is generically constructed by taking linear combinations of the $\star\phi^i$ [21].

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Then, consistency of exterior differentiation and the constancy of M_{ij} with respect to the internal coordinates implies

$$d\widetilde{\phi}^i = 0\,. \tag{4.7}$$

This in turn gives the closedness of the coassociative 4-form, in agreement with (2.19).

4.1.1 Supersymmetric Minkowski vacua

Generic supersymmetric vacua are obtained for $D_i W = 0$. This results in a negative semi-definite value of the cosmological constant at the critical point

$$V_* = -3e^K |\mathcal{W}|^2 \leq 0\,. \tag{4.8}$$

It is clear that in order to obtain a supersymmetric Minkowski vacuum the vanishing of (4.8) must be imposed, which implies $\mathcal{W} = 0$. Altogether, the conditions for supersymmetric Minkowski vacua are $\mathcal{W} = 0 = \partial_i \mathcal{W}$.

The above conditions applied to (4.1) translate into the following set of equations

$$\begin{aligned} M_{ij} t^j &= 0\,, \\ M_{ij} \tau^j &= -G_i\,, \\ \frac{1}{2} M_{ij} \tau^i \tau^j &= -g_7\,. \end{aligned} \tag{4.9}$$

The first one tells us that $t^i = \mathrm{Re}\, T^i$ should be a null eigenvector of M and that M must be therefore degenerate to have solutions. Also, there must be at least one null eigenvector, which has either only positive components, so that $t_i > 0$, or four of them positive and two negative.

Since M has reduced rank, some of the equations in the second line of (4.9) are linearly dependent. This implies that there would be some additional consistency conditions between the fluxes G_i and that some of the moduli τ_i will be left unfixed. Similarly, if a set of t_i is a null vector of M, then also λt_i is a null vector and hence there is at least one unfixed geometric modulus. These facts are in accordance with the general expectation that supersymmetric Minkowski vacua can never lead to complete moduli stabilization without taking into account non-perturbative effects.[6]

Using now the basis (4.2) and the subsequent formulas linking the quantities in the superpotential to the geometric structure of the internal manifold, we can also prove that supersymmetric Minkowski vacua are obtained if and only if the internal manifold has G_2-holonomy. Using the expression of the G_2-form $\Phi = t_i \phi^i$ and of its dual 4-form $\star\Phi = \frac{V}{t^i}\widetilde{\phi}^i$, we can compute the torsion classes taking also into account (4.6) and (4.7). On the vacuum, where (4.9) is satisfied, we find

$$d\Phi = -4\, t^i M_{ij} \widetilde{\phi}^j = 0\,, \tag{4.10}$$

$$d \star \Phi = d\left(\frac{V}{t^i}\widetilde{\phi}^i\right) = 0\,. \tag{4.11}$$

[6]We would like to thank J.P. Derendinger for discussions on this issue.

Here we have used the fact that the differential acts only on the internal coordinates, so that V and t^i are constant. The outcome is that Minkowski supersymmetric vacua require twisted 7-tori with G_2-holonomy. The converse is also true as the ϕ^i span a basis of the 3-forms, and therefore (4.10) implies the first supersymmetry condition in (4.9).

4.1.2 Supersymmetric AdS vacua

These are determined by the vanishing of the Kähler covariant derivatives of the potential (4.1):

$$D_i\mathcal{W} = M_{ij}T^j + iG_i - \frac{1}{T^i + \overline{T}^i}\mathcal{W} = 0\,. \tag{4.12}$$

It is straightforward to see from (4.12) that if $\mathcal{W}$ does not depend on one or more moduli then the only allowed supersymmetric vacua are Minkowski as $D_i\mathcal{W} = -\frac{1}{T^i+\overline{T}^i}\mathcal{W} = 0$. The consequence is that we can find AdS vacua only if the potential depends on all 7 moduli.

We can further strengthen this condition by separating the real and imaginary pieces of (4.12) as

$$M_{ij}t^j \,=\, \frac{1}{2t^i}\mathcal{W}_{\rm Re}\,, \tag{4.13}$$

$$M_{ij}\tau^j + G_i \,=\, \frac{1}{2t^i}\mathcal{W}_{\rm Im}\,, \tag{4.14}$$

where we have introduced the real and imaginary parts of the superpotential:

$$\mathcal{W}_{\rm Re} \,=\, \frac{1}{2}t^iM_{ij}t^j - \frac{1}{2}\tau^iM_{ij}\tau^j - G_i\tau^i + g_7 \tag{4.15}$$

$$\mathcal{W}_{\rm Im} \,=\, t^i\left(M_{ij}\tau^j + G_i\right). \tag{4.16}$$

Now, summing over all the imaginary parts (4.14) with coefficients $t^i \neq 0$, we obtain

$$\sum t^i{\rm Im}\,(D_i\mathcal{W}) = 0\,, \qquad \Leftrightarrow \qquad \mathcal{W}_{Im} = \frac{7}{2}\mathcal{W}_{\rm Im}\,, \tag{4.17}$$

which is obviously consistent only for

$$\mathcal{W}_{Im} = 0\,. \tag{4.18}$$

This condition is equivalent to

$$M_{ij}\tau^j + G_i = 0\,, \tag{4.19}$$

which also solves identically (4.14) when $\mathcal{W}_{Im} = 0$.

The fact that supersymmetric AdS vacua can be obtained only for real values of the superpotential at the critical point imposes further constraints on the possible matrices M. Indeed, if the quadratic part of $\mathcal{W}$ does not depend on some of the moduli, i.e. if M is zero in some of the rows, then no supersymmetric AdS critical points are allowed. This can be shown by considering (4.12) in these directions, where one finds that

$$D_i\mathcal{W} = iG_i - \frac{1}{T^i + \overline{T}^i}\mathcal{W} = 0\,, \tag{4.20}$$

implying that $\mathcal{W}$ should be purely imaginary. Then we have $\mathcal{W} = 0$ and due to (4.18) we are back at the Minkowski case.

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The conclusion is that, in order to obtain supersymmetric AdS solutions, the quadratic part of the superpotential should depend on all 7 moduli. In this case complete moduli stabilization is in principle possible, provided that the additional constraints on M coming from the Jacobi identities (2.11) are satisfied. The vacua are identified by

$$\begin{aligned} M_{ij}t^j &= \frac{1}{2t^i}\left(\frac{2}{3}G_i\tau^i - \frac{1}{6}g_7\right), \\ M_{ij}\tau^j + G_i &= 0\,. \end{aligned} \tag{4.21}$$

Once again we can use these conditions to check the form of the allowed G_2-structure. The exterior differential on the G_2-form and its dual now give

$$d\Phi = -4\,t^i M_{ij}\widetilde{\phi}^j = \frac{\lambda}{t^i}\widetilde{\phi}^i \sim \star\Phi = \frac{V}{t^i}\widetilde{\phi}^i\,, \tag{4.22}$$

$$d\star\Phi = d\left(\frac{V}{t^i}\widetilde{\phi}^i\right) = 0\,, \tag{4.23}$$

where we used (4.13), the expansion of the dual 4-form $\star\Phi$, and again the fact that we are taking the differential only with respect to the internal coordinates, so that V and t^i are constant. We see that on the vacuum the only non-zero intrinsic torsion class is $W_1 \sim \frac{\lambda}{V}$, i.e. the manifold has weak G_2-holonomy. The outcome is that supersymmetric AdS_4 vacua require twisted 7-tori with weak G_2-holonomy. Finally, we can go backwards and show that having weak G_2-holonomy implies the supersymmetry condition (4.13), and hence a supersymmetric AdS_4 solution.

4.2 Examples

In order to provide examples of vacua that satisfy the general supersymmetry conditions derived in the previous section, we have first to identify the possible matrices M satisfying the Jacobi constraints (2.11) and then analyse the corresponding superpotentials. These constraints are required to give vacua of the effective potential, which have a well defined interpretation in terms of 11-dimensional geometries. Weaker constraints, such as consistency of the effective theory, may lead to additional vacua, which however do not correspond to compactifications of M-theory. An example of this phenomenon is the AdS_4 vacuum of [5], that fixes all the moduli of the 4-dimensional theory. Although this is a consistent truncation of $\mathcal{N}=4$ gauged supergravity, it does not satisfy the 10-dimensional Jacobi identities corresponding to (2.11) [6].

The conditions (2.11) give very strong constraints on the possible terms allowed in the superpotentials. In order to derive consistent sets of τ_{IJ}^K we can use the fact that (2.11) are just the Jacobi identities for the group-manifold on which, after taking the quotient with a discrete subgroup, we compactify M-theory. This leads to the requirement that τ_{IJ}^K are the structure constants of a 7-dimensional algebra whose adjoint is identified with the fundamental of $sl(7)$ [18]. All possibilities consistent with the orbifold projection can then be analysed. It turns out that they fall in four main categories:

- $SO(p,q)\times U(1)$, for $p+q=4$,

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- $\mathrm{SO}(p,q) \ltimes \mathbb{R}^4$, for $p+q=3$,
- a 2-step nilpotent[7] (metabelian) algebra $N_{7,3}$,
- a solvable algebra $S_6 \rtimes \mathrm{U}(1)$ (which contains the flat groups of [2]).

We will see that the first and third lead to superpotentials that have a quadratic part depending on all moduli, while the others do not. This means that only these groups may lead to supersymmetric AdS vacua and complete moduli stabilization, whereas the others can only lead to supersymmetric Minkowski vacua. The last two are allowed also when taking the orbifold (2.1) to act freely.

4.2.1 $\mathrm{SO}(p,q) \times \mathrm{U}(1)$ with $p+q=4$

The generic form of the matrix M corresponding to this choice of structure constants has a block form (possibly after an index reshuffling):

$$M = \begin{pmatrix} 0_3 & A \\ {}^tA & 0_4 \end{pmatrix}, \tag{4.24}$$

where the matrix A depends on 6 parameters and reads

$$A = \begin{pmatrix} a_4\frac{a_1}{a_2} & a_5\dfrac{a_1}{a_3} & -a_6\dfrac{a_1}{a_2} & a_1 \\ a_4 & -a_5\dfrac{a_2}{a_3} & a_6 & a_2 \\ -a_4\dfrac{a_3}{a_2} & a_5 & a_6\dfrac{a_3}{a_2} & a_3 \end{pmatrix}. \tag{4.25}$$

The corresponding structure constants can be used to build the algebra $so(p,q) \times u(1)$

$$[H_a, H_b] = f_{abc} H_c\,, \qquad [N_a, N_b] = g_{abc} N_c\,, \qquad [G, H_a] = [G, N_a] = 0\,, \tag{4.26}$$

where f_{abc} and g_{abc} can be the structure constants of SU(2) or $\mathrm{SL}(2,\mathbb{R})$. This depends on the actual values of the parameters a_I, since the generators H_a, N_a and G are defined in terms of the X_I generators satisfying $[X_I, X_J] = \tau_{IJ}^K X_K$ as

$$\begin{aligned}
& H_1 \equiv \pm\frac{1}{\sqrt{a_4 a_5}} X_5 \pm \frac{1}{\sqrt{a_3 a_6}} X_6\,, \qquad H_2 \equiv \mp\frac{1}{\sqrt{a_1 a_5}} X_7 \mp \frac{a_2}{\sqrt{a_1 a_3 a_4 a_6}} X_8\,, \\
& H_3 \equiv \mp\sqrt{\frac{a_3}{a_1 a_5 a_6}} X_9 \pm \frac{1}{\sqrt{a_1 a_4}} X_{10}\,, \\
& N_1 \equiv \mp\frac{1}{\sqrt{a_4 a_5}} X_5 \pm \frac{1}{\sqrt{a_3 a_6}} X_6\,, \qquad N_2 \equiv \pm\frac{1}{\sqrt{a_1 a_5}} X_7 \mp \frac{a_2}{\sqrt{a_1 a_3 a_4 a_6}} X_8\,, \\
& N_3 \equiv \pm\sqrt{\frac{a_3}{a_1 a_5 a_6}} X_9 \pm \frac{1}{\sqrt{a_1 a_4}} X_{10}\,, \\
& G \equiv X_{11}\,.
\end{aligned} \tag{4.27}$$

[7] A Lie algebra g is called n-step nilpotent when its lower central series $g^{(k+1)} = [g^{(k)}, g]$, $g^{(0)} = g$ terminates at $g^{(n)} = 0$ while it is called solvable when its derived series $g_{(k+1)} = [g_{(k)}, g_{(k)}]$, $g_{(0)} = g$ terminates for some k. Obviously nilpotency implies solvability.

In the generic case with all 6 parameters $a_I \neq 0$, this matrix gives a superpotential which has a quadratic part depending on all moduli. This is a necessary condition for obtaining supersymmetric AdS vacua with complete moduli fixing. Furthermore, it is also degenerate, which is the necessary condition to obtain supersymmetric Minkowski vacua. Unfortunately it can be checked that it does not allow for any supersymmetric vacuum, either AdS or Minkowski. The first type of vacua can only be excluded after an explicit computation of (4.21). These imply that there is no solution unless the superpotential vanishes, which implies vanishing cosmological constant. The flat vacua are, however, also excluded because all the null eigenvectors have some vanishing components. These are unacceptable, as they represent singular values of the real parts of the moduli fields.

If some of the torsions are set to zero, we can obtain supersymmetric Minkowski vacua with partial moduli stabilization. An instance is given by the choice $a_3 = a_4 = a_5 = 0$, which represents a singular limit of the algebra defined by the generators (4.27), but is a perfectly well defined choice of the matrix (4.25) and thus of (4.24). We will see later that this case is the intersection of this group choice with the one leading to flat groups, which explains the result.

A different subgroup contained in this case is the direct product of two copies of the Heisenberg algebra. This, however, does not lead to supersymmetric vacua as well.

Finally, the choice $a_1 = 0$ and $a_3 = a_4 = a_5 = a_6 = 1$, $a_2 = -1$ gives non-supersymmetric Minkowski vacua. In this case the non-trivial commutators between the generators are

$$\begin{aligned} [X_5, X_{10}] &= X_7\,, & [X_5, X_7] &= -X_{10}\,, \\ [X_5, X_9] &= X_8\,, & [X_5, X_8] &= -X_9\,, \\ [X_6, X_8] &= X_{10}\,, & [X_6, X_{10}] &= -X_8\,, \\ [X_6, X_7] &= X_9\,, & [X_6, X_9] &= -X_7\,, \end{aligned} \tag{4.28}$$

therefore forming a flat group [2]. The group described by (4.28) constitutes actually two smaller copies of the one we analyse in 4.2.4 and we refer the reader there for more details.

4.2.2 $\mathrm{SO}(p,q) \ltimes \mathbb{R}^4$ with $p+q=3$

This choice leads to a degenerate matrix that depends on six parameters and (possibly after reshuffling of the indices) takes the form:

$$M = \begin{pmatrix} 0 & b_1\frac{b_2}{b_5} & b_1\frac{b_3}{b_5} & b_1\frac{b_4}{b_5} & b_1 & -2b_1\frac{b_4}{b_6} & 0 \\ b_1\frac{b_2}{b_5} & 0 & \frac{b_2b_3b_6}{b_4b_5} & \frac{b_2b_6}{b_5} & -2\frac{b_2b_6}{b_4} & \frac{b_2b_3b_6}{b_4b_5} & 0 \\ b_1\frac{b_3}{b_5} & \frac{b_2b_3b_6}{b_4b_5} & 0 & -2\frac{b_3b_6}{b_5} & \frac{b_3b_6}{b_4} & b_3 & 0 \\ b_1\frac{b_4}{b_5} & \frac{b_2b_6}{b_5} & -2\frac{b_3b_6}{b_5} & 0 & b_6 & b_4 & 0 \\ b_1 & -2\frac{b_2b_6}{b_4} & \frac{b_3b_6}{b_4} & b_6 & 0 & b_5 & 0 \\ -2b_1\frac{b_4}{b_6} & b_2 & b_3 & b_4 & b_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.29}$$

The underlying algebra can be understood by identifying the three $\mathrm{SO}(p,q)$ generators with X_5, X_8 and X_9, while the $\mathbb{R}^4$ is generated by linear combinations of the remaining generators with coefficients that depend on the b_I parameters.

Since the matrix M contains a line of zeros, the quadratic part of the superpotential does not depend on one modulus. Consequently, it cannot lead to supersymmetric AdS$_4$ vacua. Furthermore, supersymmetric Minkowski vacua are excluded as well since generic $b_I \neq 0$ lead to a single null eigenvector, which points in the direction 7. This means that $t_1 = \ldots = t_6 = 0$, which is once again a singular limit.

4.2.3 The 2-step nilpotent algebra $N_{7,3}$

This choice leads to a quadratic superpotential, which depends on all the moduli. The corresponding matrix M is simply chosen to have non-zero values on one specific line, for instance $M_{1i} \neq 0$, and $M_{ij} = 0$, with $i, j = 2 \ldots 7$. This means that once more it depends on six parameters. It is a 2-step nilpotent algebra because the generators can be grouped in two sets G_a, $a = 1, \ldots, 4$ and H_α, $\alpha = 1, 2, 3$, with commutator relations described as

$$[G, G] = H\,, \qquad [G, H] = 0\,, \qquad [H, H] = 0\,. \tag{4.30}$$

This clearly implies that the algebra is nilpotent as any 2 commutator operations annihilate any generator. For special choices of the parameters this algebra contains the 3- or 5-dimensional Heisenberg algebras.

Thanks to the special form of the matrix M, it is easy to prove its degeneracy and also that no supersymmetric AdS or Minkowski vacua are allowed. The conditions for supersymmetric AdS lead to the vanishing of the cosmological constant, while the null eigenvectors of the matrix M necessarily contain once more vanishing components. We also checked that no Minkowski vacua are allowed even when considering complete supersymmetry breaking.

4.2.4 Flat groups

The last possibility is given by matrices containing at least 3 lines of zeros. We will show that these include the flat groups first described in [2]. These matrices can be singled out by choosing the non-vanishing geometric fluxes to have the form $\tau_{XI}{}^{J}$, where X stands for one of the indices $5, \ldots, 11$ and I, J range over the remaining ones (the flat groups are the special subcase when these are in addition antisymmetric in IJ).

Given the orbifold projections (2.1), we select a 4×4 matrix M, which, by an appropriate choice of elements, can be made degenerate. For instance, choosing $X = 5$, the only non-trivial twists are $\tau_{56}^{11} = -k_1$, $\tau_{510}^{7} = k_2$, $\tau_{59}^{8} = k_3$, $\tau_{58}^{9} = -k_4$, $\tau_{57}^{10} = -k_5$, $\tau_{511}^{6} = k_6$. One can easily check that the Jacobi identities are true for these sets of twists for any value of the k's. The quadratic part of the superpotential then depends on 4 moduli T_2, T_3, T_4, T_5 and the corresponding matrix reads

$$M = \begin{pmatrix} 0 & k_1 & k_2 & k_3 \\ k_1 & 0 & k_4 & k_5 \\ k_2 & k_4 & 0 & k_6 \\ k_3 & k_5 & k_6 & 0 \end{pmatrix}. \tag{4.31}$$

Since the superpotential (4.31) depends on only 4 moduli and the related potential is therefore of the no-scale type [30], the only vacua to be expected are flat Minkowski space-times. Using the first condition in (4.9), it is clear that such vacua can be obtained when

$$t_1 k_1 = t_6 k_6 \,, \qquad t_4 k_4 = t_3 k_3 \,, \qquad t_5 k_5 = t_2 k_2 \,, \tag{4.32}$$

and $t_1 k_1 \pm t_2 k_2 \pm t_3 k_3 = 0$. The different signs lead to the same manifold up to reshuffling of the vielbeins. It is also clear that a rescaling of the twisting parameters can be reabsorbed in a rescaling of the size moduli. Therefore these different choices do not lead to different internal manifolds either. For this reason, in the following we focus on the case given by the flat groups of [2]. These are obtained for $k_4 = k_3$, $k_6 = k_1$ and $k_5 = k_2$ and in addition we impose $k_1 + k_2 + k_3 = 0$, so that the matrix M becomes degenerate. This choice of parameters implies that (4.31) has a null eigenvector given by $\lambda\{1,1,1,1\}$.

The matrix degeneracy has a special meaning in terms of the geometry of the corresponding internal manifold. Computing the torsions (2.15) and (2.16) for the flat group at $t_2 = t_3 = t_4 = t_5 = 1$, we obtain[8]

$$d\Phi = (k_1 + k_2 + k_3)\left(\eta^{581011} - \eta^{57911} + \eta^{5678} + \eta^{56910}\right), \qquad d \star \Phi = 0 \,, \tag{4.33}$$

which vanishes for $k_1 + k_2 + k_3 = 0$, i.e. the internal manifold has G_2-holonomy. This is not a complete surprise, since the corresponding twisted tori have vanishing Riemann tensor and therefore they are flat (hence the name flat groups).

For this reason, we expect that the potential obtained using M admits a supersymmetric Minkowski vacuum with all fluxes set to zero. Indeed, the choice $k_1 = -k_2 - k_3$ fulfills the first requirement to get to a supersymmetric Minkowski vacuum: a degenerate matrix M. Furthermore, the fluxes should satisfy $G_1 = G_6 = G_7 = G_2 + G_3 + G_4 + G_5 = 0$ as a consequence of the second equation in (4.9). As expected there is a solution with all fluxes set to zero and with the moduli taking values $t_2 = t_3 = t_4 = t_5 = \lambda > 0$ and $\tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$ while T_1, T_6, T_7 are left unfixed. This is the standard Minkowski background found upon compactification on a G_2-holonomy manifold.

The metric of the space resulting from quotients of the flat groups $\mathcal{G}$ can be derived by constructing the right-invariant vielbeins η^I, that satisfy (2.10) with the flat-group structure constants. These are obtained from the right-invariant Maurer–Cartan forms $\Omega = dgg^{-1}$, where $g \in \mathcal{G}$ is a group representative (see [31]). The vielbeins obtained from Ω are

$$\begin{aligned} &\eta^5 = dy^5 \,, && \eta^6 = dy^6 - k_1 y^{11} dy^5 \,, && \eta^7 = dy^7 - k_2 y^{10} dy^5 \,, \\ &\eta^8 = dy^8 - k_3 y^9 dy^5, && \eta^9 = dy^9 + k_3 y^8 dy^5 \,, && \eta^{10} = dy^{10} + k_2 y^7 dy^5 \,, \\ & && \eta^{11} = dy^{11} + k_1 y^6 dy^5 \,, && \end{aligned} \tag{4.34}$$

and give the twisted torus metric $ds^2 = \sum_I \eta^I \otimes \eta^I$. This space is made compact by

[8] We use the shorthand notation $\eta^{I_1 \cdots I_n} = \eta^{I_1} \wedge \cdots \wedge \eta^{I_n}$.

imposing the following identifications

$$
\begin{array}{cc}
\multicolumn{2}{c}{y^5 \sim y^5 + 1,} \\
\begin{cases} y^6 \sim y^6 + \cos\left(k_1 y^5\right) \\ y^{11} \sim y^{11} - \sin\left(k_1 y^5\right) \end{cases} & \begin{cases} y^6 \sim y^6 + \sin\left(k_1 y^5\right) \\ y^{11} \sim y^{11} + \cos\left(k_1 y^5\right) \end{cases} \\
\begin{cases} y^7 \sim y^7 + \cos\left(k_2 y^5\right) \\ y^{10} \sim y^{10} - \sin\left(k_2 y^5\right) \end{cases} & \begin{cases} y^7 \sim y^7 + \sin\left(k_2 y^5\right) \\ y^{10} \sim y^{10} + \cos\left(k_2 y^5\right) \end{cases} \\
\begin{cases} y^8 \sim y^8 + \cos\left(k_3 y^5\right) \\ y^9 \sim y^9 - \sin\left(k_3 y^5\right) \end{cases} & \begin{cases} y^8 \sim y^8 + \sin\left(k_3 y^5\right) \\ y^9 \sim y^9 + \cos\left(k_3 y^5\right), \end{cases}
\end{array}
\tag{4.35}
$$

under which the vielbeins and hence the metric are globally well-defined.

These identifications follow from the right quotient of $\mathcal{G}$ with its discrete subgroup $\Gamma = \mathcal{G}(\mathbb{Z})$. One can also construct the right-invariant vector fields

$$
\begin{aligned}
X_5 &= \partial_5 + k_1\left(y^{11}\partial_6 - y^6\partial_{11}\right) + k_2\left(y^{10}\partial_7 - y^7\partial_{10}\right) + k_3\left(y^9\partial_8 - y^8\partial_9\right), \\
X_I &= \partial_I\,, \qquad I = 6, \ldots, 11\,.
\end{aligned}
\tag{4.36}
$$

which generate the flat group algebra

$$
[X_I, X_J] = -\tau_{IJ}^{K} X_K\,. \tag{4.37}
$$

Since these vector fields are right-invariant, they are well-defined on the discrete quotient defined by the right action of Γ on $\mathcal{G}$ and generate the left action on $\mathcal{G}/\Gamma$. However, not all of them are actual isometries of the compact space $\mathcal{G}/\Gamma$. Only the generators that lie in the commutant of $\mathcal{G} \equiv \mathcal{G}(\mathbb{R})$ in $\mathcal{G}(\mathbb{Z})$ are isometries of the compact space. This implies that the generators X_I, for $I = 6, \ldots, 11$ are not Killing vectors of the compact metric. Hence only X_5 generates an isometry of the metric $ds^2 = \sum_I \eta^I \otimes \eta^I$.

The twisted torus we are describing is a fibration along y^5 of three 2-tori, parametrized by the couples of coordinates $\{y^6, y^{11}\}$, $\{y^7, y^{10}\}$, $\{y^8, y^9\}$. Moreover, the identifications in (4.35) show that this fibration is nothing but a rotation of these tori around y^5 by angles depending on k_1, k_2 and k_3. Because of this twist, translations in the coordinates of the three 2-tori are not isometries while the vector field X_5, that is the sum of the translation along y^5 with the angular shifts on the tori, is a Killing vector.

The Riemann tensor of the metric $ds^2 = \sum_I \eta^I \otimes \eta^I$ is vanishing for any value of the twist parameters k_i. This means that $\mathcal{G}/\Gamma$ is nothing but flat space, and, in the compact version, flat $\mathbb{T}^7$. This explains why the resulting effective theory yields consistent Minkowski vacua (supersymmetric and non supersymmetric). There is actually an alternative (and equivalent) description of $\mathcal{G}/\Gamma$ which reproduces explicitly the flat space metric. This can be obtained by implementing the change of coordinates

$$
\begin{aligned}
y^6 &= \sin(k_1 x^5)x^6 + \cos(k_1 x^5)x^{11}\,, & y^{11} &= \cos(k_1 x^5)x^6 - \sin(k_1 x^5)x^{11}\,, \\
y^7 &= \sin(k_2 x^5)x^7 + \cos(k_2 x^5)x^{10}\,, & y^{10} &= \cos(k_2 x^5)x^7 - \sin(k_2 x^5)x^{10}\,, \\
y^8 &= \sin(k_3 x^5)x^8 + \cos(k_3 x^5)x^9\,, & y^9 &= \cos(k_3 x^5)x^8 - \sin(k_3 x^5)x^9\,, \\
& & y^5 &= x^5\,.
\end{aligned}
\tag{4.38}
$$

The resulting euclidean metric $ds^2 = \sum_I dx^I \otimes dx^I$ has the standard identifications $x^I \sim x^I + 1$ for $I = 6, \ldots, 11$. For x^5, however, the identification reads

$$\begin{aligned}
&x^5 \sim x^5 + 1\,, \\
&x^6 \sim \cos(k_1)x^6 - \sin(k_1)x^{11}\,, \qquad && x^9 \sim \sin(k_3)x^8 + \cos(k_3)x^9\,, \\
&x^7 \sim \cos(k_2)x^7 - \sin(k_2)x^{10}\,, \qquad && x^{10} \sim \sin(k_2)x^7 + \cos(k_2)x^{10}\,, \\
&x^8 \sim \cos(k_3)x^8 - \sin(k_3)x^9\,, \qquad && x^{11} \sim \sin(k_1)x^6 + \cos(k_1)x^{11}\,.
\end{aligned} \tag{4.39}$$

This reduces to the ordinary one when the twisting coefficients k_i are chosen to be integers.[9]

This description can also be obtained by introducing the left-invariant vielbeins $\widetilde{\eta}^I$, constructed from the left-invariant Maurer-Cartan form $\widetilde{\Omega} = g^{-1}dg$, and satisfying

$$d\widetilde{\eta}^I + \frac{1}{2}\tau^I_{JK}\widetilde{\eta}^J \wedge \widetilde{\eta}^K = 0\,. \tag{4.40}$$

It should be noted that the sign in (4.40) is the opposite of (2.10). These vielbeins are given by

$$\begin{aligned}
\widetilde{\eta}^5 &= dx^5\,, & \widetilde{\eta}^6 &= \cos(k_1x^5)dx^6 - \sin(k_1x^5)dx^{11}\,, \\
\widetilde{\eta}^7 &= \cos(k_2x^5)dx^7 - \sin(k_2x^5)dx^{10}\,, & \widetilde{\eta}^8 &= \cos(k_3x^5)dx^8 - \sin(k_3x^5)dx^9\,, \\
\widetilde{\eta}^9 &= \sin(k_3x^5)dx^8 + \cos(k_3x^5)dx^9\,, & \widetilde{\eta}^{10} &= \sin(k_2x^5)dx^7 + \cos(k_2x^5)dx^{10}\,, \\
\widetilde{\eta}^{11} &= \sin(k_1x^5)dx^6 + \cos(k_1x^5)dx^{11}\,,
\end{aligned} \tag{4.41}$$

and the metric $ds^2 = \sum_I \widetilde{\eta}^I \otimes \widetilde{\eta}^I$ is precisely the flat euclidean metric, as it can be clearly seen by the fact that the left-invariant vielbeins are x^5-rotations of the ordinary straight differentials.

The global identifications in this case correspond to the left quotient $\Gamma\backslash\mathcal{G}$ of $\mathcal{G}$ by Γ. The right-action is generated by the left-invariant vectors fields

$$\begin{aligned}
\widetilde{X}_6 &= \cos(k_1x^5)\partial_6 - \sin(k_1x^5)\partial_{11}\,, & \widetilde{X}_{11} &= \sin(k_1x^5)\partial_6 + \cos(k_1x^5)\partial_{11}\,, \\
\widetilde{X}_7 &= \cos(k_2x^5)\partial_7 - \sin(k_2x^5)\partial_{10}\,, & \widetilde{X}_{10} &= \sin(k_2x^5)\partial_7 + \cos(k_2x^5)\partial_{10}\,, \\
\widetilde{X}_8 &= \cos(k_3x^5)\partial_8 - \sin(k_3x^5)\partial_9\,, & \widetilde{X}_9 &= \sin(k_3x^5)\partial_8 + \cos(k_3x^5)\partial_9\,, \\
& & \widetilde{X}_5 &= \partial_5\,,
\end{aligned} \tag{4.42}$$

and it can be checked that they indeed satisfy the flat-group algebra

$$[\widetilde{X}_I, \widetilde{X}_J] = \tau^K_{IJ}\widetilde{X}_K\,. \tag{4.43}$$

Also in this case we can see explicitly that only $\widetilde{X}_5$ is generating an isometry of $\Gamma\backslash\mathcal{G}$.

Given this change of coordinates we can now understand why the Scherk-Schwarz reduction on this twisted torus (that is just an ordinary flat torus for integer k_i) fixes some moduli. The ordinary truncation of the Kaluza-Klein spectrum expansion on the flat $\mathbb{T}^7$ is done by keeping only the zero modes of the Laplace-Beltrami operators, i.e. the moduli t_I are associated to harmonic 3-forms. Instead, the Scherk-Schwarz truncation selects a different set of moduli. These are the singlets under the action of the symmetry group

[9]We have defined the sine and cosine functions with period 1.

generated by either X_L or $\widetilde{X}_R$ according to the choice of quotient $\mathcal{G}/\Gamma$ or $\Gamma\backslash\mathcal{G}$ respectively. In terms of the Kaluza-Klein modes this implies that only some (or none) of the original zero-modes are kept, while massive modes are introduced in the truncated spectrum in a way that is consistent with the higher-dimensional equations of motion. This has to be opposed to the usual Kaluza–Klein zero-mode truncation which in general is not consistent on group manifolds or cosets [31].

In order to illustrate further the above points, let us compare the ansätze for the reduction of the metric in the two cases. The ordinary truncation of the Kaluza-Klein expansion on a generic twisted torus would be

$$ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu + (G_{IJ}(y) + g_{IJ}(x))\left(dy^I + K^I_L A^L_\mu dx^\mu\right) \otimes \left(dy^J + K^J_M A^M_\nu dx^\nu\right), \tag{4.44}$$

where $G_{IJ}(y)dy^I \otimes dy^J$ is the metric of the twisted torus and $K^I_J(y)$ are the Killing vectors of the internal isometries. The Scherk–Schwarz truncation instead is essentially an expansion around the (left-) right-invariant vielbeins

$$ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu + (\delta_{IJ} + g'_{IJ}(x))\left(\eta^I + A^I_\mu dx^\mu\right) \otimes \left(\eta^J + A^J_\nu dx^\nu\right). \tag{4.45}$$

It is clear that the usual Kaluza–Klein moduli $g_{IJ}(x)$ parametrize a different set of fluctuations from the Scherk–Schwarz moduli $g'_{IJ}(x)$. Rewriting (4.45) in a form comparable to (4.44) and since $\eta^I = U^I_J(y)dy^J$, where $U^I_J(y)$ is the twisting matrix, reveals that the Scherk-Schwarz ansatz is not a truncation to zero-modes but it rather corresponds to y-dependent fluctuations $g_{KL}(x,y) = g'_{IJ}(x)U^I_K(y)U^J_L(y)$.

For the flat group, where the twisted torus is actually flat and hence $G_{IJ} = \delta_{IJ}$, the symmetry group $\mathrm{U}(1)^7$ of the first ansatz is realised as isometries of the vacuum, whereas for the Scherk-Schwarz reduction obtained as the quotient of the non-compact group $\mathcal{G}$ one cannot find a $\mathcal{G}$-invariant ground state and only a U(1) can be preserved. This U(1) is generated by the unique Killing vector we found earlier.

Coming back to the description of the effective light degrees of freedom, we can see the mass generation in the following way. In the ordinary Kaluza-Klein reduction on $\mathbb{T}^7$, one keeps the moduli associated to the harmonic 3-forms so that the G_2-form is expanded as

$$\begin{aligned}\Phi_{\mathrm{KK}} = & -t_1 dx^5 \wedge dx^6 \wedge dx^{11} + t_2 dx^6 \wedge dx^9 \wedge dx^{10} + t_3 dx^6 \wedge dx^7 \wedge dx^8 + \\ & + t_4 dx^8 \wedge dx^{10} \wedge dx^{11} - t_5 dx^7 \wedge dx^9 \wedge dx^{11} - t_6 dx^5 \wedge dx^7 \wedge dx^{10} - \\ & - t_7 dx^5 \wedge dx^8 \wedge dx^9 \,.\end{aligned} \tag{4.46}$$

It is clear that although the metrics for the ordinary $\mathbb{T}^7$ and for the left-quotient of the flat group look formally the same, the forms that can be constructed using the straight differentials are not invariant under the action of (4.39) when the k_i are not integers. More generically, this is also true for the 3-forms appearing in the definition of the G_2-structure. Therefore we cannot use the same expansion for the Scherk-Schwarz reduction as for the truncation of the Kaluza-Klein spectrum to the zero-modes.

All the above boil down to the fact that in the Scherk–Schwarz reduction one has to define all invariant forms in terms of either the left-invariant vielbeins $\tilde{\eta}^I$ or the right-invariant ones η^I, and then promote the corresponding coefficients to spacetime dependent

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moduli fields. For example, for the left-quotient $\Gamma\backslash\mathcal{G}$ the expansion of Φ would be

$$\begin{aligned}\Phi_{\rm SS} = & -t_1\widetilde{\eta}^5\wedge\widetilde{\eta}^6\wedge\widetilde{\eta}^{11} + t_2\widetilde{\eta}^6\wedge\widetilde{\eta}^9\wedge\widetilde{\eta}^{10} + t_3\widetilde{\eta}^6\wedge\widetilde{\eta}^7\wedge\widetilde{\eta}^8 + t_4\widetilde{\eta}^8\wedge\widetilde{\eta}^{10}\wedge\widetilde{\eta}^{11} - \\ & - t_5\widetilde{\eta}^7\wedge\widetilde{\eta}^9\wedge\widetilde{\eta}^{11} - t_6\widetilde{\eta}^5\wedge\widetilde{\eta}^7\wedge\widetilde{\eta}^{10} - t_7\widetilde{\eta}^5\wedge\widetilde{\eta}^8\wedge\widetilde{\eta}^9\,.\end{aligned} \tag{4.47}$$

Only the three invariant 3-forms associated to t_1, t_6 and t_7 moduli correspond to the ones in (4.46), i.e. $\widetilde{\eta}^5\wedge\widetilde{\eta}^6\wedge\widetilde{\eta}^{11} = dx^5\wedge dx^6\wedge dx^{11}$ etc. Also, a fourth harmonic form can be obtained when $t_2 = t_3 = t_4 = t_5$ and $k_1 + k_2 + k_3 = 0$. Therefore, the corresponding fields remain massless moduli also in the Scherk–Schwarz reduction, exactly as we found in the analysis of the conditions leading to supersymmetric vacua. The remaining 3-forms contain explicit dependence on the x^5 coordinate and are associated to non-harmonic 3-forms leading to massive moduli. In general, when the twisting coefficients k_i are arbitrary, the invariant G_2 form is not closed, because the rotations of the vielbeins (4.41) with respect to the usual straight differentials yields x^5-dependent factors of the form $\cos[(k_1+k_2+k_3)x^5]$ or $\sin[(k_1+k_2+k_3)x^5]$ multiplying the usual harmonic forms.

As far as the metric decomposition is concerned, one can easily see that while ordinary toroidal Kaluza-Klein compactifications associate moduli to coordinate rescalings and, therefore, to the differentials $R_I dx^I$, the Scherk-Schwarz reductions associate moduli to the vielbeins $R_I\eta^I$. The overall effect is similar to that responsible for the generation of masses in the 1-dimensional Scherk-Schwarz reduction of a field $\varphi(x,y)$ which possesses a global U(1) invariance $\varphi\to \mathrm{e}^{i\alpha}\varphi$ [1]. In that case, the Scherk–Schwarz expansion

$$\varphi(x,y) = \mathrm{e}^{i\alpha y}\sum_n \mathrm{e}^{iny}\varphi_n(x) \tag{4.48}$$

differs from the ordinary Kaluza-Klein by an additional factor depending on the parameter α. Since an integer α corresponds to a relabeling of the modes $\varphi_n(x)$, we should restrict $\alpha\in[0,1)$. This way the effective mass is always smaller than the Kaluza-Klein masses. In our flat group example the twisting coefficients k_i are the analogues of the parameter α. In the special instance that these coefficients are integers, the identifications become those of the ordinary $\mathbb{T}^7$ and one could use the ordinary Kaluza-Klein reduction, thereby getting seven massless moduli. On the other hand, the Scherk-Schwarz truncation keeps only modes which are singlets under the symmetry group generated by either X_L or $\widetilde{X}_R$, thereby excluding all massless fluctuations of the internal metric that are not invariant.

The upshot is that the Scherk-Schwarz truncation correctly captures the low-energy degrees of freedom only when the twisting coefficients are non-integers. Presumably this is also true for more general twisted tori and not just for the flat groups we considered here. However, although in ordinary supergravity the deformation parameters corresponding to the coefficients τ_{IJ}^K can be arbitrarily small, string theory usually forces some quantization conditions [31]. Then, the efficiency of this moduli stabilization scheme would be questionable.

From this discussion, it becomes once more apparent why these examples allow for non-supersymmetric vacua. Given that the manifold described by (4.34) is just flat space, it is a valid internal manifold for Minkowski vacua for any value of the twisting parameters k_i.

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Indeed, even in the case where the values of k_i do not allow for a supersymmetric Minkowski vacuum, they still allow a non-supersymmetric one. The potential is of the no-scale type and the minima are the Minkowski vacua obtained at the same values of the moduli as those of the supersymmetric critical points of the cases analysed above. The interesting fact is that, despite being flat space, this choice of twisting parameters implies that the holonomy group is not in G_2, because $d\Phi \neq 0$. This should have been expected as it is well-known that G_2-holonomy implies supersymmetry. The actual holonomy group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is now embedded in SO(7), but not in G_2, because of the twists.

The last part of the discussion on the possible vacua for the flat group manifolds is about the possibility of obtaining vacua with 4-form flux turned on. We can actually show that more supersymmetric Minkowski vacua could be found by turning on the 4-form flux in the directions containing the index 5, which means turning on precisely the linear terms in (3.9) that depend on the 4 moduli T_2, T_3, T_4, T_5. When this happens the potential gets new terms from the 4-form fluxes, which may fix the axions to a non-zero value, while the volume moduli remain fixed at the same values as before. This happens, for instance, when generating a potential from fluxes of the form $-i(k_2T_2 + k_3T_3 + k_1T_5)$.

Since the corresponding internal manifold still has G_2-holonomy, this vacuum should be inconsistent as an 11-dimensional configuration because of the no-go theorem of [46]. What really happens is that this solution can still be considered a consistent background, though trivially related to the previous one. This is due to the fact that the fluxes that are turned on in this way correspond to trivial fluxes on 4-forms, which are not dual to the 4-cycles when the twisting is performed. As noted above, the introduction of the τ_{IJ}^K parameters and the choice of performing a Scherk-Schwarz reduction imply that we expand around non-harmonic 3-forms. Actually, the 4-form flux leading to supersymmetric vacua is cohomologically zero. For example, the flux corresponding to the superpotential $-i(k_2T_2 + k_3T_3 + k_1T_5)$ is $g = -4(k_2\widetilde{\phi}^2 + k_3\widetilde{\phi}^3 + k_1\widetilde{\phi}^5) = 4d\phi^4$, as we can see by using $\widetilde{\phi}^i = \frac{t_i^2}{V}(\star\phi^i)$.

In the previous reduction we distinguished the non-trivial part g and the trivial one, which is expressed as τC and simply shifts the values of the axions. If, on the other hand, we let g also have a trivial part, then the consistent backgrounds are generically given by $G = g + \tau C$, which is harmonic. In our example this is the case because G is identically vanishing. The punchline is that this vacuum is indistinguishable from the previous one, from the 11-dimensional point of view. Also in the effective theory, the appearance of this type of flux in the gauge algebra and in the lagrangian can be removed entirely by field redefinitions [31].

If one introduces (cohomologically) non-trivial fluxes instead, then one does not find any vacuum but rather a run-away potential, in accordance with the no-go theorem [45, 46]. This shows that the above vacuum, as well as many of those presented in the literature, for instance those of [38], are a trivial redefinition of the no-flux vacua.

The last comment concerns the possible corrections to these backgrounds due to higher order derivative terms in the action. Looking at R^4 terms, there are four types of possible corrections (see for instance [47, 48]). Since the Riemann tensor is exactly vanishing on the vacuum, we can see that these vacua are stable against these corrections. When varying the contribution of the R^4 terms to the potential with respect to the moduli, one still gets a

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critical point as its variation vanishes for any value of the moduli that was already unfixed and without changing the value of the ones that were fixed before. It would be of course very interesting to reproduce explicitly these contributions as a modification to the $\mathcal{N}=1$ superpotential.

5. Type IIA superpotentials

We consider now the circle reduction of the potential (3.19) to type IIA theory. In the first subsection we assume that (3.19) has general validity and we perform the reduction for a generic internal manifold X_7 with an isometry. In the second subsection we reduce the twisted 7-torus when the choice of structure constants allows isometries.

5.1 General reduction

For a general fibered G_2-structure, the manifold X_7 is defined as a circle fibration of an SU(3)-structure manifold M_6, with a metric of the form [29]

$$ds^2_X = \alpha \otimes \alpha + \pi^* ds^2_M \,, \tag{5.1}$$

where π defines the fibration. The reduction is performed by expressing the G_2-structure in terms of the 6-dimensional SU(3)-structure as

$$\begin{aligned} \Phi &= (\pi^* J) \wedge \alpha + (\pi^* \rho)\,, & (5.2)\\ \star\Phi &= \frac{1}{2}(\pi^* J) \wedge (\pi^* J) + (\pi^* \hat{\rho}) \wedge \alpha\,. & (5.3) \end{aligned}$$

Here ρ and $\hat{\rho}$ are the real and imaginary parts of the $(3,0)$-form Ω.

In our case the 1-form α reads

$$\alpha = e^{\frac{2}{3}\phi} \left(dy^{11} + A + \sigma\right), \tag{5.4}$$

with $d\sigma = f_{(2)}$, the background Ramond–Ramond (RR) 2-form flux of type IIA theory. Notice that σ should not be globally defined on M_6, since we would like a flux that is not pure gauge. In addition, π is defined so that the metric of the 6-dimensional base is in the string frame. In other words:

$$ds^2_7 = e^{-\frac{2\phi}{3}} ds^2_6 + e^{\frac{4\phi}{3}} (dy^{11} + C_1)^2 \,, \tag{5.5}$$

and we conclude that $\pi^* J = e^{-\frac{2}{3}\phi} J$ and $\pi^* \rho = e^{-\phi}\rho$. Finally, the 11-dimensional fields and background fluxes are reduced accordingly:

$$C = \mathcal{C} + e^{-\frac{2}{3}\phi} B \wedge \alpha, \quad g = f_{(4)} + e^{-\frac{2}{3}\phi} h \wedge \alpha\,. \tag{5.6}$$

Plugging these expressions in (3.19) yields

$$4\mathcal{W}_{IIA} = \frac{1}{2}\int J_c \wedge J_c \wedge (dA + f_{(2)}) + \int f_{(4)} \wedge J_c - \int (dJ_c + h) \wedge \Omega_c \,, \tag{5.7}$$

where we have defined

$$\Omega_c = \mathcal{C} + ie^{-\phi}\rho\,, \qquad J_c = B + iJ\,. \tag{5.8}$$

The 10-dimensional dilaton ϕ is related to the 4-dimensional one ϕ_4 as

$$\phi = \phi_4 - \frac{1}{4}\log\det g_{(6)}\,, \tag{5.9}$$

where $g_{(6)}$ is the metric of M_6.

The expression (5.7) of the type IIA superpotential agrees with those presented in [35, 20, 6, 9, 41] upon appropriate redefinitions of the SU(3)-structure. Furthermore, in type IIA the superpotential can be further completed with terms cubic in the moduli by considering the massive IIA theory. Since there is no known lift of massive type IIA supergravity to 11-dimensional supergravity we cannot expect similar terms to arise from the reduction of (3.19). Notice that in [49] massive type IIA *superstrings* were obtained from M-theory by a Scherk-Schwarz reduction on a $\mathbb{T}^3$ that shrinks to zero size. Unfortunately, this mechanism lies outside the region of validity of our effective supergravity approach.

5.2 Reduction of the twisted 7-torus

The superpotentials we derived from twisted tori compactifications of M-theory have the same form as the type IIA superpotentials obtained from twisted tori compactifications. They differ, however, in the number of independent terms. M-theory yields a 4-dimensional $\mathcal{W}$ as in (3.9), which contains more quadratic couplings between the moduli than those found in the type IIA superpotentials. This implies that either these couplings have to vanish when considering reductions to type IIA supergravity from M-theory or that the resulting type IIA backgrounds are not twisted tori. These possibilities depend on the choices of isometries upon which one reduces the 11-dimensional theory.

We have seen in the previous section that only some of the symmetries of the Scherk-Schwarz ansatz for the reduction on twisted tori are actual isometries of the possible internal metrics. If, for instance, we look at the 2-step nilpotent algebra $N_{7,3}$, we can obtain non-trivial 2-form fluxes, with a constant dilaton, upon reducing along any of its symmetries. All the symmetries generating this algebra become isometries of the possible metrics (though none of them is a consistent vacuum of the effective theory). This means that upon reducing to type IIA along these isometries one would find a constant dilaton $\phi(x,y) = \phi(x)$ and possibly a non-trivial 2-form flux. This implies that the reduced space is a twisted torus; indeed, this can be explicitly verified.

For this type of reductions, we can identify the 7 moduli $T_i, i = 1,\ldots,7$ of X_7 with the type IIA ones of [5] and compare the respective superpotentials. In [5], superpotentials for a type IIA orientifold of $\mathbb{T}^6/(\mathbb{Z}_2\times\mathbb{Z}_2)$ with both physical and geometrical fluxes were constructed from consistent $\mathcal{N}=1$ truncations of gauged $\mathcal{N}=4$ supergravity. Reduction of X_7 along an isometry which gives a constant dilaton reproduces their setup. We now fix y^{11} as the Killing direction and require that $\tau_{11I}^{J} = 0$ in order to have a constant dilaton.

The starting point for connecting with the results of [5] is to consider our 7-dimensional twisted toroidal orbifold as a fibration of the form (5.1), where the base is a twisted 6-dimensional orbifold $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The latter is naturally equipped with an SU(3)-structure specified by J and Ω. Because of our conventions for the G_2 3-form (2.2), the SU(3) decomposition reads

$$\Phi = e^{-\frac{2}{3}\phi} J \wedge \alpha + e^{-\phi}\, \hat{\rho}\,, \tag{5.10}$$

with the SU(3)-structure being defined as

$$J = t_1' dy^5 \wedge dy^6 + t_2' dy^7 \wedge dy^8 + t_3' dy^9 \wedge dy^{10}\,, \tag{5.11}$$

$$\Omega = \rho + i\hat{\rho} = -\sqrt{\frac{t_1' t_2' t_3'}{u_1 u_2 u_3}} (u_1 dy^5 + i dy^6) \wedge (u_2 dy^7 + i dy^8) \wedge (u_3 dy^9 + i dy^{10})\,. \tag{5.12}$$

These SU(3)-structure forms are normalized as $\Omega \wedge \overline{\Omega} = -\frac{4i}{3} J \wedge J \wedge J$, so that the corresponding G_2-structure (5.10) is also properly normalized. As usual, t_i' and u_i are the Kähler and complex structure moduli respectively, in terms of which the non-vanishing components of the metric are

$$g_{55} = t_1' u_1\,, \qquad g_{66} = \frac{t_1'}{u_1}\,, \qquad g_{77} = t_2' u_2\,, \qquad g_{88} = \frac{t_2'}{u_2}\,, \qquad g_{99} = t_3' u_3\,, \qquad g_{1010} = \frac{t_3'}{u_3}\,. \tag{5.13}$$

The superpotential of [5] depends on 7 chiral multiplets (T_A', U_A', S'), for $A = 1, 2, 3$, whose real parts are (t_A', u_A', s') with

$$s' = \sqrt{\frac{s}{u_1 u_2 u_3}}\,, \qquad u_1' = \sqrt{\frac{s u_2 u_3}{u_1}}\,, \qquad u_2' = \sqrt{\frac{s u_1 u_3}{u_2}}\,, \qquad u_3' = \sqrt{\frac{s u_1 u_2}{u_3}}\,. \tag{5.14}$$

Here s is the 4-dimensional dilaton defined as $s \equiv e^{-2\phi_4} = e^{-2\phi}(t_1' t_2' t_3')$.

A comparison with the G_2 form

$$\begin{aligned} \Phi = {} & (t_1 dy^5 \wedge dy^6 - t_2 dy^7 \wedge dy^8 - t_3 dy^9 \wedge dy^{10}) \wedge dy^{11} - \\ & - t_4 dy^6 \wedge dy^7 \wedge dy^9 + t_5 dy^6 \wedge dy^8 \wedge dy^{10} + t_6 dy^5 \wedge dy^7 \wedge dy^{10} + t_7 dy^5 \wedge dy^8 \wedge dy^9 \end{aligned} \tag{5.15}$$

yields the following identification of moduli:

$$t_1 = t_1'\,, \quad t_2 = -t_2'\,, \quad t_3 = -t_3'\,, \quad t_4 = u_1'\,, \quad t_5 = s'\,, \quad t_6 = -u_3'\,, \quad t_7 = -u_2'\,. \tag{5.16}$$

This obviously extends to the full chiral multiplets. Turning on the geometric fluxes amounts to replacing dy^I by η^I and hence the identifications remain the same.

The decomposition of fluxes works straightforwardly. The allowed 4-form fluxes in 11-dimensions

$$g_{78910}\,, \qquad g_{56910}\,, \qquad g_{5678}\,, \tag{5.17}$$

$$g_{581011}\,, \qquad g_{671011}\,, \qquad g_{57911}\,, \qquad g_{68911}\,, \tag{5.18}$$

reduce to 4- and 3-form fluxes in type IIA:

$$f_{78910}\,, \qquad f_{56910}\,, \qquad f_{5678}\,, \tag{5.19}$$

$$h_{5810}\,, \qquad h_{6710}\,, \qquad h_{579}\,, \qquad h_{689}\,. \tag{5.20}$$

The geometrical fluxes of the form τ^{11}_{IJ} correspond to 2-form RR fluxes since $\eta^{11} = dy^{11} + \sigma \Rightarrow d\eta^{11} = f_{(2)} = \frac{1}{2}\tau^{11}_{IJ}\eta^I \wedge \eta^J$. For the twisted 7-torus, these are

$$f_{56}\,, \qquad f_{78}\,, \qquad f_{910}\,. \tag{5.21}$$

Finally, the 7-form flux $g_{567891011}$ obviously reduces to a 6-form flux

$$f_{5678910}\,. \tag{5.22}$$

We can now re-write the M-theory superpotential (3.9), using the identifications (5.16):[10]

$$\begin{aligned} 4\mathcal{W}_{IIA} = {} & f_{5678910} + i(T_1 f_{78910} + T_2 f_{56910} + T_3 f_{5678}) + iSh_{579} - \\ & - i(U_1 h_{5810} + U_2 h_{6710} + U_3 h_{689}) - (T_1T_2 f_{910} + T_2T_3 f_{56} + T_3T_1 f_{78}) + \\ & + S(T_1\tau^6_{79} + T_2\tau^8_{95} + T_3\tau^{10}_{57}) - (T_1U_1\tau^6_{810} + T_2U_2\tau^8_{106} + T_3U_3\tau^{10}_{68}) + \\ & + (T_1U_2\tau^5_{710} + T_1U_3\tau^5_{89} + T_2U_1\tau^7_{105} + T_2U_3\tau^7_{96} + T_3U_1\tau^9_{58} + T_3U_2\tau^9_{67})\,. \end{aligned} \tag{5.23}$$

The above expression contains all the terms of the superpotential of [5][11] except for the cubic one, since this corresponds to a massive type IIA reduction, which, as we said earlier, should be invisible in our approach. The two superpotentials match, as we have seen, because $\tau^J_{11I} = 0$.

Assuming a similar identification of the type IIA moduli fields as above, the terms in (3.9) involving τ^J_{11I} would mean that couplings of the form SU and UU appear in the IIA superpotential. These are not present in [5] and therefore we should expect them when the type IIA theory is compactified on an internal manifold which is not a twisted torus. It would be interesting to give a concrete example of this phenomenon, but as this requires a further analysis of the various quotient group manifolds presented here we postpone it for future work.

6. Summary and outlook

In this paper we studied the Scherk-Schwarz reduction of M-theory in the presence of fluxes, using the equivalence between Scherk-Schwarz reductions and compactifications on twisted tori. The latter allows us to go beyond the realm of exceptional holonomy, in this case G_2, in a concrete and controllable framework. In particular, the 11-dimensional supergravity action can be explicitly reduced and we can obtain the 4-dimensional effective potential for the light modes. This result is the sum of the expressions (3.2), (3.4) and (3.6). Now, the fact that our models have precisely a G_2-structure implies that the 4-dimensional theory is an $\mathcal{N} = 1$ supergravity and hence that the potential should be derivable by a $\mathcal{N} = 1$ superpotential. Indeed, the superpotential (3.9) reproduces precisely the potential we found from the explicit reduction, using also the Kähler potential (3.8). Furthermore, the Kähler

[10]The superfields here correspond to the primed ones in (5.16) and are precisely those that appear in [5].
[11]The sign difference in the terms involving the geometrical fluxes is due to our sign convention in (2.10).

potential and superpotential can be written in terms of the geometric structure of the 7-manifold, in the compact form (3.18) and (3.19) respectively. These are in agreement and extend expressions that have been proposed previously.

The analysis of the vacua is hindered by the fact that the coefficients of the superpotential depend on the structure constants that define the twisting, and the latter satisfy the highly non-trivial constraints (2.11). Hence, we first presented some generic features concerning the vacuum structure of superpotentials of the form (4.1). This form is the most general one that can be obtained from M-theory compactifications keeping only the leading order terms in the 11-dimensional supergravity action. We found that the matrix that determines the quadratic part has to be degenerate in order to have supersymmetric Minkowski vacua and that it has to yield dependence on all moduli in order to have supersymmetric AdS vacua. Notice that the latter condition is stronger than the fact that the superpotential has to depend on all moduli in order to have supersymmetric AdS solutions. We were able also to show that, in agreement with the no-go theorem of [45, 46], supersymmetric Minkowski/AdS solutions can be obtained only from twisted 7-tori with G_2/weak G_2-holonomy unless sources are introduced or higher order derivative corrections are considered.

Subsequently we presented some classes of solutions to the constraints (2.11). Despite the wealth of solutions, the best that can be achieved are supersymmetric and non-supersymmetric Minkowski vacua with at most 3 moduli stabilized. However, our study revealed several interesting properties of this type of reductions. Among others, a geometric way of understanding the mechanism behind moduli fixing in Scherk-Schwarz compactifications was obtained, using the flat groups as toy models. As a by-product, the difference between ordinary Kaluza–Klein compactification on a twisted torus (which would yield seven massless moduli for a flat group) and Scherk-Schwarz reduction (which yields four massless moduli) was elucidated. The underlying principle is that the expansion of all physical fields in internal pieces and external moduli fields should respect a certain symmetry group. The choice of this group selects the fields that appear in the effective action and in some cases extra care is required in order to identify correctly the massless degrees of freedom.

Another point concerns the introduction of fluxes. Although it is a priori assumed that the background fluxes are harmonic, in cases where the internal manifold is a different parametrization of flat space the flux evaluated at the solution is vanishing. Again this is a consequence of the no-go theorem. Also, it can be seen explicitly from our supersymmetry condition $M_{ij}\tau^j + G_i = 0$, which fixes the axion values in terms of the 4-form fluxes. This relation, valid for both Minkowski and AdS vacua, once expressed using differential forms and the relations between M_{ij} and τ_{IJ}^K, implies the exactness of the allowed 4-form fluxes. Consequently, since by assumption the 4-form fluxes G_i are harmonic, they are identically zero.

A consistency check of our approach is performed by reducing our results to type IIA theory. In this way we were able to derive both the generic form of the type IIA superpotential for a compactification on a manifold with SU(3)-structure and also the superpotentials of [5]. The latter were derived in a complementary bottom-up approach, which utilized the machinery of $\mathcal{N} = 4$ supergravity and its $\mathcal{N} = 1$ truncations. Our top-to-bottom approach hints towards the existence of more quadratic couplings between the type IIA moduli than those found in [5], when a non-trivial dilaton is present and the internal 6-manifolds are not twisted tori. A more precise analysis of these geometries is left for future work.

Another interesting open problem in this context is to establish a dictionary between the effective 4-dimensional approach and the one we have followed here (including also sources). It would be worthwhile, for example, to understand the correspondence between the constraints imposed on the structure constants of the $\mathcal{N} = 1$ gauging by the Jacobi identities and the 10- or 11-dimensional constraints involving the geometrical and physical fluxes (the analogues of (2.11) and (3.5)). This will clarify whether the techniques based on gauged supergravity are able to capture all possible effective supergravities and, vice versa, the extent to which the $\mathcal{N} = 1$ gauged supergravities actually admit a 10- or 11-dimensional space–time interpretation.

Understanding the deformations of G_2-structures is another urgent task that will help us obtain a better picture of the landscape of $\mathcal{N} = 1$ M-theory vacua. As we have already pointed out, the Kähler potential (3.18) is the Hitchin functional on the space of stable 3-forms. It has been argued that a similar Hitchin functional, which appears as the Kähler potential for type II compactifications on 6-manifolds with SU(3)-structure, describes the moduli space of SU(3)-structures [41]. It would be extremely interesting to extend the analysis of [41] for 7-manifolds with G_2-structure.

Finally, it would be important to extend the considerations of this paper to other concrete examples of manifolds with G_2-structure. In particular, one can consider freely acting G_2-orbifolds [26] which are not plagued by extra blow-up moduli. Analyzing the possibilities for moduli stabilization due to Scherk-Schwarz deformations and form fluxes in these examples is left for future work.

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A. Dual gauged supergravity algebras from the M-theory pseudo-action

As we have seen in section 3, the pseudo-action (3.10) can be very useful to describe part of the degrees of freedom of the 3-form, using the dual 6-form as well as introducing the dual form-fluxes. Here we will see, for the general case of Scherk-Schwarz compactifications of M-theory on $\mathbb{T}^7$, how this formalism let us derive the effective gauged supergravity algebras in the dual form of [50, 52], without the need of cumbersome group-theoretical arguments. This means that we can naturally integrate out the massive tensor degrees of freedom appearing in the standard compactifications.

For this purpose we will continue to follow the convention on the splitting of the 11-dimensional indices in space-time ones $\mu, \nu \dots$ and internal ones $I, J, K, \dots$. Also, in order to describe effectively the 4-dimensional degrees of freedom without introducing massive tensor fields, we have to choose an appropriate set of connections and curvatures to be plugged in the pseudo-action (3.10).

The main point in this choice is that consistency implies that not all vectors can be in the **21** of $sl(7)$, i.e. described by the $C_{\mu IJ}$ connection; some of them (up to seven), those which will become massive by eating the scalars dual to the tensor fields, should be described by the dual connection $A_{\mu IJKLM}$. For this reason one has to split the couples of indices IJ into those belonging to closed forms (hence related to the 2-cycles $\mathcal{C}_2$), and the orthogonal ones. We will see the reasons for this choice in the following. These conditions will turn out to be consistency conditions required in order to make the formalism work. Since we do not want tensor fields to appear naked in the effective theory coming from the pseudo-action, we choose to turn on the curvatures

$$G_{\mu\nu IJ}\,, \qquad G_{\mu IJK}\,, \qquad G_{IJKL}\,, \qquad \widetilde{G}_{\mu\nu IJKLM}\,, \qquad \widetilde{G}_{\mu IJKLMN}\,, \qquad \widetilde{G}_{IJKLMNP}\,, \tag{A.1}$$

and the dual connections

$$C_{\mu\nu\rho}\,, \qquad C_{\mu\nu I}\,, \qquad C_{\mu IJ}\,, \qquad A_{\mu\nu\rho\sigma IJ}\,, \qquad A_{\mu\nu\rho IJK}\,, \qquad A_{\mu\nu IJKL}\,, \qquad A_{\mu IJKLM}\,, \tag{A.2}$$

where for the moment we leave the possibility of having both the standard vectors $C_{\mu IJ}$ and the dual ones $A_{\mu IJKLM}$.

The variation of the (3.10) pseudo-action with respect to (A.2) gives

$$\partial_\sigma \widetilde{G}_{IJKLMNP} - \frac{7}{2}\tau_{[IJ}^{Q}\widetilde{G}_{\sigma KLMNP]Q} + 70 G_{\sigma[IJK}G_{LMNP]} = 0\,, \tag{A.3}$$

$$\partial_{[\rho}\widetilde{G}_{\sigma]IJKLMN} + 20 G_{[\rho[IJK}G_{\sigma]LMN]} + 15 G_{\rho\sigma[IJ}G_{KLMN]} = 0\,, \tag{A.4}$$

$$\partial_{[\nu}\widetilde{G}_{\rho\sigma]IJKLM} + 20 G_{[\nu[IJK}G_{\rho\sigma]LM]} = 0\,, \tag{A.5}$$

$$\tau_{[IJ}^{P}G_{KLM]P} = 0\,, \tag{A.6}$$

$$\partial_\mu G_{IJKL} - 6\tau_{[IJ}^{P}G_{\mu KL]P} = 0\,, \tag{A.7}$$

$$\partial_{[\rho}G_{\sigma]IJK} - \frac{3}{2}\tau_{[IJ}^{P}G_{\rho\sigma K]P} = 0\,, \tag{A.8}$$

$$\partial_{[\nu}G_{\rho\sigma]IJ} = 0\,. \tag{A.9}$$

We can start solving them from the last one, inserting each time the solution in the previous equation. In this way, we find that the curvatures should be defined by

$$G_{\mu\nu IJ} = 2\partial_{[\mu}C_{\nu]IJ}\,, \tag{A.10}$$

$$G_{\mu IJK} = \partial_\mu C_{IJK} + 3\tau_{[IJ}^{P}C_{\mu K]P}\,, \tag{A.11}$$

$$G_{IJKL} = g_{IJKL} + 6\tau_{[IJ}^{P}C_{KL]P}\,, \tag{A.12}$$

$$
\begin{aligned}
\tau^P_{[IJ}g_{KLMN]} &= 0\,, && \text{(A.13)}\\
\widetilde{G}_{\mu\nu IJKLM} &= 2\partial_{[\mu}A_{\nu]IJKLM} - 60C_{[\mu[IJ}\tau^P_{KL}C_{\nu]M]P} - 40C_{[\mu[IJ}\partial_{\nu]}C_{KLM]}\,, && \text{(A.14)}\\
\widetilde{G}_{\mu IJKLMN} &= \partial_\mu A_{IJKLMN} + 15\,\tau^P_{[IJ}A_{\mu KLMN]P} - 20C_{[IJK}\partial_\mu C_{LMN]} - \\
&\quad - 30\,C_{\mu[IJ}\left(g_{KLMN]} + 6\tau^P_{KL}C_{MN]P}\right), && \text{(A.15)}\\
\widetilde{G}_{IJKLMNP} &= \tilde{g}_{IJKLMNP} + 21\tau^Q_{[IJ}A_{KLMNP]Q} - 70C_{[IJK}\left(g_{LMNP]} + 3\tau^Q_{LM}C_{NP]Q}\right), && \text{(A.16)}
\end{aligned}
$$

and are subject to the constraint

$$\tau\left(C_{[\mu[IJ}\partial_\nu C_{\rho]KL]}\right) = 0\,. \tag{A.17}$$

This is a very strong constraint on the allowed vector fields. It appears because we are varying with respect to an unconstrained set of $C_{\mu IJ}$. On the other hand, we know that only a subset of them should be dualized. This set can be chosen precisely in a way which gets rid of the constraint (A.17). By splitting $C_{\mu IJ} = \tau^K_{IJ}C_{\mu K} + C^0_{\mu IJ}$ [52], we see that we do not get the constraint (A.17) when varying the pseudo-action (3.10) with respect to $C_{\mu K}$, as this term appears in a total derivative on the internal compact space. This is precisely what is required by the dualization procedure. The (up to 7) vectors $C_{\mu K}$ have to be dualized, while the $C^0_{\mu IJ}$ components appear in the final action.

From these solutions we can see two important consequences for the effective theory. The first one is that there are no space-time tensor fields left. The standard degrees of freedom of the 7 tensor fields $C_{\mu\nu I}$ are replaced by the dual 7 scalars $A^P \equiv \epsilon^{PIJKLMN}A_{IJKLMN}$. It is also explicitly made clear here that these are in the conjugate representation of SL(7) with respect to the original ones (if the tensors are in the **7**, the dual scalars are in the $\overline{\mathbf{7}}$). Also, the 21 vector fields are split in two sets (as explained above). A number $n \leq 7$ of them are described by the dual vectors coming from the 6-form potential $\tilde{A}^Q_\mu = \epsilon^{IJKLMNP}\tau^Q_{IJ}A_{\mu KLMNP}$, whereas the remaining $21-n$ are described by the original 3-form $C_{\mu IJ}$. This is the only way to close consistently the above Bianchi identities. If all the independent components of the vectors $C_{\mu IJ}$ are expanded on the 2-cycles, then one can solve directly the Bianchi identities (A.3)–(A.9) without the need for introducing the dual potentials.

The second important aspect is given by the relation (A.13). Since the constant fluxes of the 4-form fields act as coupling constants in the effective theory, this relation implies a change in the structure constants of this theory. At this stage we can plug these solutions back in the original action and therefore also complete the effective theory gauged supergravity algebra. As in [18, 31] this can be obtained by looking at the commutators of the gauge transformations of the vector fields as they are in a faithful representation of the gauge group. These transformations read

$$
\begin{aligned}
\delta G^I_\mu &= \partial_\mu\omega^I - \tau^I_{JK}\omega^J G^K_\mu\,,\\
\delta C_{\mu IJ} &= \partial_\mu\Sigma_{IJ} + 3G^P_\mu\tau^S_{[PI}\Sigma_{J]S} - 2\omega^Q\tau^S_{Q[I}C_{\mu J]S} - \omega^R g_{RIJQ}G^Q_\mu\,,
\end{aligned}
$$

$$\delta A_{\mu IJKLM} = \partial_\mu \Theta_{IJKLM} - 6G_\mu^P \tau_{[PI}^S \Theta_{JKLM]S} + 5\omega^Q \tau_{Q[I}^S A_{\mu JKLM]S} + \\ + \omega^R g_{RIJKLMQ} G_\mu^Q - 10\Sigma_{[IJ}\partial_\mu C_{KLM]} - 10C_{\mu[IJ}\omega^Q g_{KLM]Q} - \\ - 12\Sigma_{[IJ}\left(2g_{KLMP]} + 3\tau_{MP}^S C_{KLS}\right) G_\mu^P - 10C_{[IJK}\omega^P g_{LM]NP} G_\mu^P \,. \quad \text{(A.18)}$$

The algebra can thus be obtained by assigning the generators

$$T_A = \{Z_I, W^{IJ}, W_{IJ}\}\,, \quad \text{(A.19)}$$

matching the parameters $\{\omega^I, \Sigma_{IJ}, \epsilon^{IJKLMNP}\Theta_{KLMNP}\}$ respectively. The commutators of these generators give

$$\begin{aligned} [Z_I, Z_J] &= \tau_{IJ}^K Z_K + g_{IJKL} W^{KL} + g_{IJKLMNP}\epsilon^{KLMNPQR} W_{QR}\,, \\ [Z_I, W^{JK}] &= 2\tau_{IM}^{[J} W^{K]M} - 10\epsilon^{JKLMNPQ} g_{ILMN} W_{PQ}\,, \\ [Z_I, W_{JK}] &= 2\tau_{I[J}^P W_{K]P} = \tau_{JK}^P W_{PI}\,, \\ [W^{IJ}, W^{KL}] &= 30\, W_{PQ}\epsilon^{MNPQIJ[K}\tau_{MN}^{L]}\,, \\ [W^{IJ}, W_{KL}] &= [W_{IJ}, W_{KL}] = 0\,, \end{aligned} \quad \text{(A.20)}$$

and they constitute the gauge algebra in the dual formulation, which was also obtained in a similar form using group-theoretical arguments in [52]. The closure of the algebra can be obtained by recalling the split in the vectors, which removes some problematic terms in the commutators of the gauge transformations of $C_{\mu IJ} = C^0_{\mu IJ}$. Also, one has to use some remarkable relations, first found in [52], among the algebra generators : $\tau_{[IJ}^P W_{K]P} = 0$ and $10\,\epsilon^{ABCDIJK} g_{ABCD} W_{JK} = -2\tau_{JK}^I W^{JK}$.

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